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# Characteristic foliations of material evolution: from remodeling to aging

**Víctor Manuel Jiménez**
*Departamento de Física y Matemáticas, Universidad de Alcalá (UAH), Alcalá de Henares, Madrid, Spain*

**Manuel de León**
*Real Academia de Ciencias Exactas, Fisicas y Naturales;*
*Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), Madrid, Spain*

**Marcelo Epstein**
*Department of Mechanical and Manufacturing Engineering, University of Calgary, Calgary, Alberta, Canada*

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**Abstract**

For any body-time manifold $\mathbb{R}\times\mathcal{B}$ there exists a groupoid, called the *material groupoid*, encoding all the material properties of the material evolution. A smooth distribution, the *material distribution*, is constructed to deal with the case in which the material groupoid is not a Lie groupoid. This new tool provides a unified framework to deal with general non-uniform material evolution.

**Keywords**

Lie groupoid, uniformity, material groupoid, material evolution, remodeling

## 1. Introduction

In this paper, we use the approach developed by W. Noll [1, 2] based on the notion of the so-called material diffeomorphism between pairs of points in the body, namely, a map between the respective tangent spaces that renders the constitutive responses identical. In Noll's terminology, a body is said to be materially uniform if all of its points are mutually materially isomorphic. In a uniform body, a smooth field of material isomorphisms is nothing but a distant parallelism whose integrability is equivalent to the homogeneity of the body. The existence of material symmetries works like a gauge of freedom for these parallelisms, so that the geometric notion associated with the body is a so-called $G$-structure, where $G$ is a model for the group of material symmetries. In this context, local homogeneity is equivalent to the integrability of the associated $G$-structure. However, this approach depends on the choice of a linear frame (some material archetype) and, moreover, it cannot be used for non-uniform bodies [3].

The natural extension of the theory was the consideration of a more general kind of algebraic/geometric structures, namely Lie groupoids. The collection of all the material isomorphisms of a body constitutes always

**Corresponding author:**
Víctor Manuel Jiménez, Departamento de Física y Matemáticas, Universidad de Alcalá (UAH), Av. de León, 4A, 28805 Alcalá de Henares, Madrid, Spain.
Email: victormanuel.jimenez@uah.es

a groupoid, called the material groupoid. This point of view has been developed extensively in the book [5] (see also [6–8]) even for more general bodies where the material groupoid is not a Lie groupoid.

One of the contributions of this paper is the introduction of a new space–time framework for the construction of groupoids canonically associated with the time evolution of a material body [9–11]. This has been done to some extent in previous papers, but we now consider a more general scenario that includes non-uniform bodies under non-uniform evolutions. This technique will permit us to define a type of global remodeling of non-uniform bodies (Definition 17) and a definition of differentiable global aging (Definition 23).

Recall that the infinitesimal approximation of a Lie groupoid is its Lie algebroid, just as the tangent space to a manifold at a point is the linear approximation of a neighborhood of the point. One of the constructions we have developed in several previous papers is that of the *characteristic distribution* associated with a subgroupoid of a Lie groupoid, even when the subgroupoid is not differentiable. This construction is a generalization of the notion of the Lie algebroid associated with a Lie groupoid [5, 6].

The characteristic distributions associated with material groupoids give rise to the respective foliations. Uniform aging is also presented for the first time in this paper. It expresses the fact that, although the body ages, all material points age in the same way over time.

To incorporate the time dimension brought about by the physical reality of phenomena of material evolution, we extend the notion of a material manifold $\mathcal{B}$ to the so-called body-time manifold, namely, the fiber bundle $\mathcal{C} = \mathbb{R} \times \mathcal{B}$ over $\mathbb{R}$. The embeddings are now fiber bundle embeddings into the trivial fiber bundle $\mathbb{R} \times \mathbb{R}^3$ over $\mathbb{R}$; such an embedding $\Phi$ is called a *history*, because for a given point $X_0$ in $\mathcal{B}$, $\Phi_t(X_0)$ describes the history of the material point. The crucial point now is to consider the vertical subbundle, $\mathcal{V}$, associated with the body-time manifold $\mathcal{C}$, and the associated frame groupoid $\Phi(\mathcal{V}) \rightrightarrows \mathcal{C}$. This groupoid will play a similar role to that played by the 1-jets groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$ on $\mathcal{B}$ for the case of non-evolving simple elastic materials (see Section 3). This permits to define the corresponding material groupoid $\Omega(\mathcal{C})$ for a given constitutive law as a subgroupoid of $\Phi(\mathcal{V})$, consisting just of those material isomorphisms, connecting not only material points but also different instants of time. Thus, we may consider a temporal counterpart of uniformity called remodeling. Indeed, a material particle $X \in \mathcal{B}$ undergoes a remodeling process when all the points at $\mathbb{R} \times \{X\}$ are connected by material isomorphisms; if this happens for all the material points, then $\mathcal{C}$ presents a global remodeling process. In other words, the intrinsic material properties of the body do not change in time. Particular cases are the phenomena of growth and resorption (remodeling with mass increase or mass decrease of the material body). This kind evolution may be found in biological tissues [12] or Wolff's law of trabecular architecture of bones [13]).

Any phenomenon of material evolution other than pure remodeling can be considered as an aging process. In other words, a material particle $X \in \mathcal{B}$ undergoes aging if not all the instants are connected by a material isomorphism and, therefore, the intrinsic material response is not preserved in time. Our approach will allow us to introduce the concept of smooth aging.

From a more technical standpoint, some of the most relevant results contained in the paper are as follows.

**Corollary 21.**
*An initially uniform body undergoes smooth uniform remodeling if, and only if, $\Omega(\mathcal{C})$ is a transitive Lie subgroupoid of $\Phi(\mathcal{V})$.*

**Theorem 22.**
*Let $\mathcal{C}$ be a body-time manifold. The body-material foliation $\mathcal{F}$ (respectively, uniform material foliation $\mathcal{G}$) divides $\mathcal{C}$ into maximal smooth uniform remodeling processes (respectively, uniform remodeling processes).*

**Theorem 23.**
*Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ undergoes a smooth uniform remodeling process (respectively, uniform remodeling) if, and only if,* $\dim\left(A\Omega(\mathcal{C})^{\sharp}_{(t,X)}\right) = 4$ *(respectively,* $\dim\left(A\Omega(\mathcal{C})^{B}_{(t,X)}\right) = 4$*) for all instants $t$ and all particles $X$, with $A\Omega(\mathcal{C})^{\sharp}_{(t,X)}$ (respectively, $A\Omega(\mathcal{C})^{B}_{(t,X)}$) the fiber of $A\Omega(\mathcal{C})^{\sharp}$ (respectively, $A\Omega(\mathcal{C})^{B}$) at $(t, X)$.*

Roughly speaking, Corollary 21 establishes that the uniform differentiable remodeling is equivalent to $\Omega(\mathcal{C})$ being a transitive Lie subgroupoid of $\Phi(\mathcal{V})$. This clarifies the difference between remodeling and uniform differentiable remodeling. Theorem 22 proves that there are two maximal foliations of $\mathcal{C}$ separating the evolution of the material into uniform remodeling and smooth uniform remodelings, respectively. On the other hand, Theorem 23 shows that, using the material distribution, one can imagine the shape of the foliation associated with the differentiable uniform remodeling by calculating the dimensions of its leaves. In particular, if the dimension

is four at any material point, the evolution has a uniform and differentiable remodeling. Thus, studying whether the evolution has a uniform and differentiable remodeling is reduced to the study of the linear equation (41).

On the other hand, Theorem 24 determines the material foliation by uniformly differentiable components of the body at each instant. As one can imagine, in Proposition 25 it is shown that if one freezes the evolution in the leaves given by Theorem 22, the leaves of Theorem 24 are recovered. Finally, Theorem 26 is the analog of Theorem 23 for differentiable remodeling, thus giving a *computational condition* (linear equation) for studying differentiable remodeling.

Owing to the length of this paper, it is important to note that Sections 2 and 3 consist mainly of preliminaries to make the text as self-contained as possible. Indeed, Section 2 introduces the basic concepts of groupoids, following [14, 15]. It also includes a construction that is essential throughout the paper, the so-called *characteristic distribution*, which was introduced for the first time in [6]. Section 3 is devoted to a quick introduction to the theory of simple materials and the concept of uniformity. A first use of the characteristic distribution is given in this section following [7, 8, 16] (see also the book [5]). Finally, Section 4.1 is devoted to the introduction of the concept of material evolution in a very abstract setting. Readers familiar with these topics could skip Sections 2 and 3 and go directly to the last three sections of the paper where the new results are described.

## 2. Groupoids and distributions

### 2.1. Groupoids

We start with a very brief introduction on *(Lie) groupoids* which turns out to be crucial to understand the results shown in this paper. Groupoids were introduced in 1926 by Brandt [17] as a natural generalization of groups. With the addition of a differential structure we obtain the notion of *Lie groupoid*, which was first introduced by Ehresmann in a series of articles [4, 18–20] and redefined by Pradines in [21].

We follow the most relevant reference on groupoids [14]. In [22, 23] we can find a more intuitive view of this topic. The book [15] (in Spanish) is also recommendable as a rigorous introduction to groupoids.

**Definition 1.** Let $M$ be a set. A *groupoid* over $M$ is given by a set $\Gamma$ provided with the maps $\alpha, \beta : \Gamma \to M$ (*source map* and *target map*, respectively), $\epsilon : M \to \Gamma$ (*section of identities*), $i : \Gamma \to \Gamma$ (*inversion map*) and $\cdot : \Gamma_{(2)} \to \Gamma$ (*composition law*) where for each $k \in \mathbb{N}$, $\Gamma_{(k)}$ is given by $k$ points $(g_1, \ldots, g_k) \in \Gamma \times \overset{(k)}{\ldots} \times \Gamma$ such that $\alpha(g_i) = \beta(g_{i+1})$ for $i = 1, \ldots, k-1$. It satisfies the following properties:

(1) The maps $\alpha$ and $\beta$ are surjective and for each $(g, h) \in \Gamma_{(2)}$,

$$\alpha(g \cdot h) = \alpha(h), \quad \beta(g \cdot h) = \beta(g).$$

(2) Associativity of the composition law, i.e.,

$$g \cdot (h \cdot k) = (g \cdot h) \cdot k, \ \forall (g, h, k) \in \Gamma_{(3)}.$$

(3) For all $g \in \Gamma$,

$$g \cdot \epsilon(\alpha(g)) = g = \epsilon(\beta(g)) \cdot g.$$

Therefore,

$$\alpha \circ \epsilon \circ \alpha = \alpha, \quad \beta \circ \epsilon \circ \beta = \beta.$$

As $\alpha$ and $\beta$ are surjective, we have that

$$\alpha \circ \epsilon = Id_M, \quad \beta \circ \epsilon = Id_M,$$

where the map $Id_M$ is the identity map at $M$.

(4) For each $g \in \Gamma$,

$$i(g) \cdot g = \epsilon(\alpha(g)), \quad g \cdot i(g) = \epsilon(\beta(g)).$$

Then,

$$\alpha \circ i = \beta, \quad \beta \circ i = \alpha.$$

These maps are called *structure maps*. The usual notation for a groupoid is $\Gamma \rightrightarrows M$.

Here $M$ is denoted by $\Gamma_{(0)}$ and it is identified with the set $\epsilon(M)$ of identities of $\Gamma$; $\Gamma$ is also denoted by $\Gamma_{(1)}$. The elements of $M$ are called *objects* and the elements of $\Gamma$ are called *morphisms*. Furthermore, for each $g \in \Gamma$ the element $i(g)$ is called *inverse of* $g$ and it is denoted by $g^{-1}$.

**Definition 2.** Let $\Gamma \rightrightarrows M$ be a groupoid. The map $(\alpha, \beta) : \Gamma \to M \times M$ is called the *anchor map*. The space of sections of the anchor map is denoted by $\Gamma_{(\alpha,\beta)}(\Gamma)$.

Roughly speaking, a groupoid may be thought as a set of "*arrows*" ($\Gamma$) joining points ($M$) together with a composition law with similar rules to the composition of maps.

**Definition 3.** If $\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ are two groupoids, then a *morphism of groupoids* from $\Gamma_1 \rightrightarrows M_1$ to $\Gamma_2 \rightrightarrows M_2$ consists of two maps $\Phi : \Gamma_1 \to \Gamma_2$ and $\phi : M_1 \to M_2$ satisfying the commutative relations of the following diagrams:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\Phi} & \Gamma_2 \\ {\scriptstyle\alpha_1}\uparrow & & \downarrow{\scriptstyle\alpha_2} \\ M_1 & \xrightarrow{\phi} & M_2 \end{array} \qquad \begin{array}{ccc} \Gamma_1 & \xrightarrow{\Phi} & \Gamma_2 \\ {\scriptstyle\beta_1}\uparrow & & \downarrow{\scriptstyle\beta_2} \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

$$\begin{array}{ccc} (\Gamma_1)_{(2)} & \xrightarrow{\cdot} & \Gamma_1 \\ \downarrow{\scriptstyle\Phi_{(2)}} & & \downarrow{\scriptstyle\Phi} \\ (\Gamma_2)_{(2)} & \xrightarrow{\cdot} & \Gamma_2 \end{array}$$

where

$$\Phi_{(2)}(g_1, h_1) = (\Phi(g_1), \Phi(h_1))$$

for all $(g_1, h_1) \in (\Gamma_1)_{(2)}$. Equivalently, for any $g_1 \in \Gamma_1$

$$\alpha_2(\Phi(g_1)) = \phi(\alpha_1(g_1)), \quad \beta_2(\Phi(g_1)) = \phi(\beta_1(g_1)), \tag{1}$$

where $\alpha_i$ and $\beta_i$ are the source and the target maps of $\Gamma_i \rightrightarrows M_i$, respectively, for $i = 1, 2$, and preserves the composition, i.e.,

$$\Phi(g_1 \cdot h_1) = \Phi(g_1) \cdot \Phi(h_1), \ \forall (g_1, h_1) \in \Gamma_{(2)}.$$

We will denote this morphism as $\Phi$.

An immediate consequence of this definition is that $\Phi$ preserves the identities, i.e.,

$$\Phi \circ \epsilon_1 = \epsilon_2 \circ \phi,$$

where $\epsilon_i$ is the section of identities of $\Gamma_i \rightrightarrows M_i$ for $i = 1, 2$.

Using the notion of morphism of groupoids, we may define a *subgroupoid* of a groupoid $\Gamma \rightrightarrows M$ as a groupoid $\Gamma' \rightrightarrows M'$ such that $M' \subseteq M$, $\Gamma' \subseteq \Gamma$ and the inclusion map is a morphism of groupoids. More explicitly, $\Gamma'(\subseteq \Gamma) \rightrightarrows M'(\subseteq M)$ is a subgroupoid of $\Gamma \rightrightarrows M$ if it is a groupoid with the same structure maps as $\Gamma$.

**Example 1.** A group is a groupoid over a point. Indeed, let $G$ be a group and let $e$ be the identity element of $G$. Then, $G \rightrightarrows \{e\}$ is a groupoid, where the operation law of the groupoid, $\cdot$, is the operation in $G$.

**Example 2.** For any set $A$, we shall consider the product space $A \times A$. Then, the maps:

$$\alpha(a,b)=a,\ \ \beta(a,b)=b,\ \forall(a,b)\in A\times A;$$
$$(c,b)\cdot(a,c)=(a,b),\ \forall(c,b),(a,c)\in A\times A;$$
$$\epsilon(a)=(a,a),\ \forall a\in A;$$
$$(a,b)^{-1}=(b,a),\ \forall(a,b)\in A\times A;$$

endow $A\times A$ with the structure of a groupoid over $A$, called the *pair groupoid*.

Observe that, if $\Gamma\rightrightarrows M$ is an arbitrary groupoid over $M$, then the anchor map $(\alpha,\beta):\Gamma\to M\times M$ is a morphism from $\Gamma\rightrightarrows M$ to the pair groupoid of $M$.

Next, let us give the key example of a groupoid in this paper.

**Example 3.** Let us consider a vector bundle $\pi:A\to M$ on a manifold $M$. For each $z\in M$, denote by $A_z$ the fiber of $A$ over $z$. Then, $\Phi(A)$ is the set of linear isomorphisms $L_{x,y}:A_x\to A_y$, for $x,y\in M$ and it may be endowed with the structure of a groupoid with the following structure maps:

(i) $\alpha\left(L_{x,y}\right)=x$;
(ii) $\beta\left(L_{x,y}\right)=y$;
(iii) $L_{y,z}\cdot G_{x,y}=L_{y,z}\circ G_{x,y}$, $L_{y,z}:A_y\to A_z$, $G_{x,y}:A_x\to A_y$.

This groupoid is called the *frame groupoid on A*. A particular relevant case arises when we choose $A$ equal to the tangent bundle $TM$ of $M$. In this latter case, the groupoid will be called a *1-jets groupoid on M* and is denoted by $\Pi^1(M,M)$. Note that any isomorphism $L_{x,y}:T_xM\to T_yM$ may be written as a 1-jet $j^1_{x,y}\psi$ of a local diffeomorphism $\psi$ from $M$ to $M$ such that $\psi(x)=y$. Remember that the 1-jet $j^1_{x,y}\psi$ is given by the induced tangent map $T_x\psi:T_xM\to T_yM$. To study in detail the formalism of 1-jets, see [24].

**Definition 4.** Let $\Gamma\rightrightarrows M$ be a groupoid with $\alpha$ and $\beta$ the source map and target map, respectively. For each $x\in M$, the set

$$\Gamma^x_x=\beta^{-1}(x)\cap\alpha^{-1}(x),$$

is called the *isotropy group of* $\Gamma$ at $x$. The set

$$\mathcal{O}(x)=\beta\left(\alpha^{-1}(x)\right)=\alpha\left(\beta^{-1}(x)\right),$$

is called the *orbit* of $x$, or *the orbit* of $\Gamma$ through $x$.

Note that the orbit of a point $x$ consists of the points which are "*connected*" with $x$ by a morphism in the groupoid whereas the isotropy group is given by the morphisms connecting $x$ with $x$. Of course, the composition law is globally defined inside the isotropy groups. Thus, the isotropy groups inherit a *bona fide* group structure.

**Definition 5.** If $\mathcal{O}(x)=\{x\}$ or, equivalently, $\beta^{-1}(x)=\alpha^{-1}(x)=\Gamma^x_x$, then $x$ is called a *fixed point*. *The orbit space of* $\Gamma$ is the space of orbits of $\Gamma$ on $M$. If $\mathcal{O}(x)=M$ for all $x\in M$ (or, equivalently, $(\alpha,\beta):\Gamma\to M\times M$ is a surjective map) the groupoid $\Gamma\rightrightarrows M$ is called *transitive*. If every $x\in M$ is a fixed point, then the groupoid $\Gamma\rightrightarrows M$ is called *totally intransitive*. Furthermore, a subset $N$ of $M$ is called *invariant* if it is a union of some orbits.

Finally, the sets,

$$\alpha^{-1}(x)=\Gamma_x,\quad \beta^{-1}(x)=\Gamma^x,$$

are called an *$\alpha$-fiber at x* and *$\beta$-fiber at x*, respectively.

**Definition 6.** Let $\Gamma\rightrightarrows M$ be a groupoid. We may define the *left translation on* $g\in\Gamma$ as the map $L_g:\Gamma^{\alpha(g)}\to\Gamma^{\beta(g)}$, given by

$$h\mapsto g\cdot h.$$

We may define the right translation on $g$, $R_g:\Gamma_{\beta(g)}\to\Gamma_{\alpha(g)}$ analogously.

Note that the identity map on $\Gamma^x$ may be written as the following translation map,

$$Id_{\Gamma^x}=L_{\epsilon(x)}. \tag{2}$$

For any $g\in\Gamma$, the left (respectively, right) translation on $g$, $L_g$ (respectively, $R_g$), is a bijective map with inverse $L_{g^{-1}}$ (respectively, $R_{g^{-1}}$).

Different kinds of structures may be imposed on a groupoid. In particular, we are interested in the so-called *Lie groupoids* which are endowed with a differentiable structure.

**Definition 7.** A *Lie groupoid* is a groupoid $\Gamma \rightrightarrows M$ such that $\Gamma$ is a smooth manifold, $M$ is a smooth manifold and the structure maps are smooth. Furthermore, the source and the target map are submersions.

A *Lie groupoid morphism* is a groupoid morphism which is differentiable. An *embedding of Lie groupoids* is a Lie groupoid morphism $(\Phi, \phi)$ such that $\Phi$ and $\phi$ are embeddings. A *Lie subgroupoid* of $\Gamma \rightrightarrows M$ is a Lie groupoid $\Gamma' \rightrightarrows M'$ such that $\Gamma'$ and $M'$ are submanifolds of $\Gamma$ and $M$, respectively, and the inclusion maps $i_{\Gamma'} : \Gamma' \hookrightarrow \Gamma$ $i_{M'} : M' \hookrightarrow M$ become a morphism of Lie groupoids. Here $\Gamma' \rightrightarrows M'$ is said to be a *reduced Lie subgroupoid* if it is transitive and $M' = M$.

It is not difficult to check that if there exists a reduced Lie subgroupoid of a groupoid $\Gamma \rightrightarrows M$, then $\Gamma \rightrightarrows M$ is transitive.

Note that the following statements are immediate:

- $\epsilon$ is an injective immersion;
- for each $g \in \Gamma$, the left translation $L_g$ (respectively, right translation $R_g$) is a diffeomorphism, for all $g \in \Gamma$;
- for each $k \in \mathbb{N}$, $\Gamma_{(k)}$ is a smooth manifold, for all $k \in \mathbb{N}$;
- the $\beta$-fibers and the $\alpha$-fibers are closed submanifolds of $\Gamma$.

**Example 4.** A Lie group is a Lie groupoid over a point.

**Example 5.** Let $M$ be a manifold. The pair groupoid $M \times M \rightrightarrows M$ is a Lie groupoid.

**Example 6.** The frame groupoid $\Phi(A)$ on a vector bundle $A$ is a Lie groupoid (see Example 3). Indeed, let $\left(x^i\right)$ and $\left(y^j\right)$ be local coordinates on open neighborhoods $U, V \subseteq M$ and $\{\alpha_p\}$ and $\{\beta_q\}$ be local basis of sections of $A_U$ and $A_V$, respectively. The corresponding local coordinates $\left(x^i \circ \pi, \alpha^p\right)$ and $\left(y^j \circ \pi, \beta^q\right)$ on $A_U$ and $A_V$ are given by:

- for all $a \in A_U$,
$$a = \alpha^p(a)\, \alpha_p\left(x^i(\pi(a))\right);$$
- for all $a \in A_V$,
$$a = \beta^q(a)\, \beta_q\left(y^j(\pi(a))\right).$$

Then, a local coordinate system on $\Phi(A)$ may be constructed as

$$\Phi\left(A_{U,V}\right) : \left(x^i, y^j_i, y^j_i\right),$$

where $A_{U,V} = \alpha^{-1}(U) \cap \beta^{-1}(V)$ and for each $L_{x,y} \in \alpha^{-1}(x) \cap \beta^{-1}(y) \subseteq \alpha^{-1}(U) \cap \beta^{-1}(V)$:

- $x^i\left(L_{x,y}\right) = x^i(x)$;
- $y^j\left(L_{x,y}\right) = y^j(y)$;
- $y^j_i\left(L_{x,y}\right) = A_{L_{x,y}}$, where $A_{L_{x,y}}$ is the matrix associated with the induced map of $L_{x,y}$ by the local coordinates $\left(x^i \circ \pi, \alpha^p\right)$ and $\left(y^j \circ \pi, \beta^q\right)$.

In the particular case of the 1-jets groupoid on $M$, $\Pi^1(M, M)$, the local coordinates are denoted as

$$\Pi^1(U, V) : \left(x^i, y^j, y^j_i\right), \tag{3}$$

where, for each $j^1_{x,y}\psi \in \Pi^1(U, V)$:

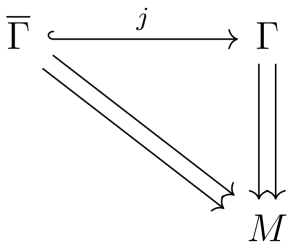
- $x^i\left(j^1_{x,y}\psi\right) = x^i(x)$;
- $y^j\left(j^1_{x,y}\psi\right) = y^j(y)$;
- $y^j_i\left(j^1_{x,y}\psi\right) = \dfrac{\partial\left(y^j \circ \psi\right)}{\partial x^i_{|x}}$.

The most important example of groupoid in this paper is the *material groupoid* which will be constructed as a subgroupoid of special cases of the frame groupoid. In particular, we deal with the 1-jets groupoid $\Pi^1(\mathcal{B}, \mathcal{B})$ on a manifold $\mathcal{B}$ (body) and a frame groupoid $\Phi(\mathcal{V})$ of the vertical bundle $\mathcal{V}$ of a given vector bundle $\mathcal{C}$ (material evolution).

### 2.2. *Characteristic distribution*

From now on, we consider the following elements: $\Gamma \rightrightarrows M$ *is a Lie groupoid and* $\overline{\Gamma}$ *is a subgroupoid of* $\Gamma$ *(not necessarily a Lie subgroupoid of* $\Gamma$*) over the same manifold* $M$.

We also denote by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\epsilon}$, and $\overline{i}$ the restrictions of the structure maps $\alpha$, $\beta$, $\epsilon$, and $i$ of $\Gamma$ to $\overline{\Gamma}$ (see the following diagram).




Here $j$ is the inclusion map. Thus, we construct the so-called *characteristic distribution* $A\overline{\Gamma}^T$ (see [5, 6]).

A (local) vector field $\Theta \in \mathfrak{X}_{loc}(\Gamma)$ on $\Gamma$ is called *admissible* for the couple $\left(\Gamma, \overline{\Gamma}\right)$ if it satisfies that:

(i) $\Theta$ is tangent to the $\beta$-fibers,

$$\Theta(g) \in T_g\beta^{-1}(\beta(g)),$$

for all $g$ in the domain of $\Theta$;

(ii) $\Theta$ is invariant by left translations,

$$\Theta(g) = T_{\epsilon(\alpha(g))}L_g(\Theta(\epsilon(\alpha(g)))),$$

for all $g$ in the domain of $\Theta$;

(iii) the (local) flow $\varphi_t^{\Theta}$ of $\Theta$ satisfies

$$\varphi_t^{\Theta}(\epsilon(x)) \subseteq \overline{\Gamma},$$

for all $x \in M$.

Thus, roughly speaking, an admissible vector field is a left invariant vector field on $\Gamma$ whose flow at the identities is totally contained in $\overline{\Gamma}$. We denotes the family of admissible vector fields for the couple $\left(\Gamma, \overline{\Gamma}\right)$ by $\mathcal{C}_{(\Gamma,\overline{\Gamma})}$ or simply $\mathcal{C}$ if there is no danger of confusion.

Then, for each $g \in \Gamma$, $A\overline{\Gamma}_g^T$ is the vector subspace of $T_g\Gamma$ linearly generated by the evaluation of the admissible vector fields at $g$. Observe that, for all $g \in \Gamma$, the zero vector $0_g \in T_g\Gamma$ is contained in the fiber of the distribution at $g$, namely $A\overline{\Gamma}_g^T$ (we refer to [5, 6, 16] for non-trivial examples). Furthermore, it satisfies that a vector field $\Theta$ of $\Gamma$ fulfills conditions (i) and (ii) if, and only if, its local flow $\varphi_t^{\Theta}$ is left-invariant or, equivalently,

$$L_g \circ \varphi_t^{\Theta} = \varphi_t^{\Theta} \circ L_g, \ \forall g, t.$$

Therefore, condition (iii) is equivalent to the following:

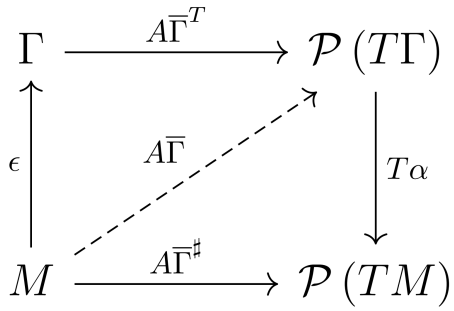
(iii′) the (local) flow $\varphi_t^{\Theta}$ of $\Theta$ at $\overline{g}$ is totally contained in $\overline{\Gamma}$, for all $\overline{g} \in \overline{\Gamma}$.

Thus, the admissible vector fields are the left-invariant vector fields on $\Gamma$ whose integral curves are confined inside or outside $\overline{\Gamma}$.

The distribution $A\overline{\Gamma}^T$ generated by the vector spaces $A\overline{\Gamma}_g^T$ is called *characteristic distribution of* $\overline{\Gamma}$. As an immediate result, we have that this distribution is *differentiable*.

*Remark 1.* This construction of the characteristic distribution associated with a subgroupoid $\overline{\Gamma}$ of a Lie groupoid $\Gamma$ may be thought as a generalization of the construction of the associated Lie algebroid to a given Lie groupoid (see [14]).

The algebraic structure associated with a groupoid allows us to define more objects. In particular, one of them is a smooth distribution over the base $M$ denoted by $A\overline{\Gamma}^{\sharp}$. The other is a "*differentiable*" correspondence $A\overline{\Gamma}$ which associates to any point $x$ of $M$ a vector subspace of $T_{\epsilon(x)}\Gamma$. Both constructions are characterized by the following diagram.




Here $\mathcal{P}(E)$ defines the power set of $E$. Therefore, for any $x \in M$, the fibers are characterized by

$$A\overline{\Gamma}_x = A\overline{\Gamma}^T_{\epsilon(x)},$$
$$A\overline{\Gamma}^\sharp_x = T_{\epsilon(x)}\alpha\left(A\overline{\Gamma}_x\right).$$

The distribution $A\overline{\Gamma}^\sharp$ is called the *base-characteristic distribution of* $\overline{\Gamma}$.

Note that, taking into account that $A\overline{\Gamma}^T$ is locally generated by left-invariant vector fields, we have that for each $g \in \Gamma$,

$$A\overline{\Gamma}^T_g = T_{\epsilon(\alpha(g))}L_g\left(A\overline{\Gamma}^T_{\epsilon(\alpha(g))}\right),$$

i.e., the characteristic distribution is *left-invariant*.

**Theorem 1.** (de León et al. [5] and Jiménez et al. [6]) *Let $\Gamma \rightrightarrows M$ be a Lie groupoid and $\overline{\Gamma}$ be a subgroupoid of $\Gamma$ (not necessarily a Lie groupoid) over $M$. Then, the characteristic distribution $A\overline{\Gamma}^T$ is integrable and its associated foliation $\overline{\mathcal{F}}$ of $\Gamma$ satisfies that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{F}}$.*

This result is a consequence of the celebrated Stefan–Sussman theorem [25, 26] which deals with the integrability of singular distributions.

Thus, the distribution $A\overline{\Gamma}^T$ is the tangent distribution of a smooth (possibly) singular foliation $\overline{\mathcal{F}}$. Each leaf at a point $g \in \Gamma$ is denoted by $\overline{\mathcal{F}}(g)$. Furthermore, the family of the leaves of $\overline{\mathcal{F}}$ at points of $\overline{\Gamma}$ is called the *characteristic foliation of* $\overline{\Gamma}$. Note that the leaves of the characteristic foliation covers $\overline{\Gamma}$, but it is not exactly a foliation of $\overline{\Gamma}$ ($\overline{\Gamma}$ is not necessarily a manifold). The foliation $\overline{\mathcal{F}}$ satisfies the following.

(i) For any $g \in \Gamma$,

$$\overline{\mathcal{F}}(g) \subseteq \Gamma^{\beta(g)}.$$

Indeed, if $g \in \overline{\Gamma}$, then

$$\overline{\mathcal{F}}(g) \subseteq \overline{\Gamma}^{\beta(g)}.$$

(ii) For any $g, h \in \Gamma$ such that $\alpha(g) = \beta(h)$, we have

$$\overline{\mathcal{F}}(g \cdot h) = g \cdot \overline{\mathcal{F}}(h).$$

In this way, without any assumption of differentiability over $\overline{\Gamma}$, we have that $\overline{\Gamma}$ is a union of leaves of a foliation of $\Gamma$. This provides some kind of "*differentiable*" structure over $\overline{\Gamma}$. The following result provides us an intuition about the maximality condition of the characteristic foliation.

**Corollary 2.** *Let $\overline{\mathcal{H}}$ be a foliation of $\Gamma$ such that $\overline{\Gamma}$ is a union of leaves of $\overline{\mathcal{H}}$ and*

$$\overline{\mathcal{H}}(g) \subset \Gamma^{\beta(g)}.$$

*Then, the characteristic foliation $\overline{\mathcal{F}}$ is coarser that $\overline{\mathcal{H}}$, i.e.,*

$$\overline{\mathcal{H}}(g) \subseteq \overline{\mathcal{F}}(g), \quad \forall g \in \Gamma. \tag{4}$$

**Proof.** The result follows from the facts that $\overline{\mathcal{H}}$ is generated by left-invariant vector fields and any of these left-invariant vector field $\Theta \in T\overline{\mathcal{H}}$ is obviously tangent to the characteristic distribution. □

As a consequence, *the fibers* $\overline{\Gamma}^x$ *are submanifolds of* $\Gamma$ *for all* $x \in M$ *if, and only if,* $\overline{\Gamma}^x = \overline{\mathcal{F}}(\epsilon(x))$ *for all* $x \in M$.

**Proposition 3 (Consistency).** *Let* $\Gamma \rightrightarrows M$, $\Gamma' \rightrightarrows M'$ *be two Lie groupoids and let* $\Phi : \Gamma \to \Gamma'$ *be an embedding of Lie groupoids. Consider a (not necessarily Lie) subgroupoid* $\overline{\Gamma}$ *of* $\Gamma$. *Then, the image of the characteristic foliation* $\overline{\mathcal{F}}$ *of* $\overline{\Gamma}$ *by* $\Phi$ *is the characteristic foliation of* $\Phi\left(\overline{\Gamma}\right)$ *as a subgroupoid of* $\Gamma'$.

**Proof.** First, note that $\Phi(\Gamma)$ is a Lie groupoid of $\Gamma'$ on $\phi(M)$, where $\phi$ is the projection of $\Phi$ on the base manifolds, because $\Phi$ is an embedding of Lie groupoids.

Let $\Theta \in \mathfrak{X}_{loc}(\Gamma)$ be an admissible vector field for the couple $\left(\Gamma, \overline{\Gamma}\right)$, i.e.:

•$\Theta$ is left-invariant;
•the (local) flow $\varphi_t^{\Theta}$ of $\Theta$ satisfies

$$\varphi_t^{\Theta}(\epsilon(x)) \subseteq \overline{\Gamma},$$

for all $x \in M$.

Then, the pushforward $\Phi_*\Theta$ is an admissible vector field for the couple $\left(\Phi(\Gamma), \Phi\left(\overline{\Gamma}\right)\right)$. In fact, because $\Phi$ is a morphism of Lie groupoids, we have that $\Phi_*\Theta$ is left-invariant.

On the other hand, the (local) flow of $\Phi_*\Theta$ is given by $\Phi \circ \varphi_t^{\Theta} \circ \Phi^{-1}$, where $\varphi_t^{\Theta}$ is the local flow of $\Theta$. Thus, at each $x = \phi(y) \in \phi(M)$, the local flow of $\Phi_*\Theta$ at the identity on $y$, $\Phi \circ \varphi_t^{\Theta}(\epsilon(x))$ is totally contained in $\Phi(\Gamma)$, i.e.,

$$\Phi \circ \varphi_t^{\Theta}(\epsilon(x)) \in \Phi\left(\overline{\Gamma}\right), \quad \forall t.$$

Analogously, given an admissible vector field $\Lambda$ for the couple $\left(\Phi(\Gamma), \Phi\left(\overline{\Gamma}\right)\right)$, the pushforward $\Phi_*^{-1}\Lambda$ is an admissible vector field for the couple $\left(\Gamma, \overline{\Gamma}\right)$. Hence, we have proved that the image of the characteristic foliation $\overline{\mathcal{F}}$ of $\overline{\Gamma}$ by $\Phi$ is the characteristic foliation of $\Phi\left(\overline{\Gamma}\right)$ as a subgroupoid of $\Phi(\Gamma)$.

Finally, owing to the fact that $\Phi(\Gamma)$ is a Lie subgroupoid of $\Gamma'$ and $\Phi\left(\overline{\Gamma}\right)$ is contained in $\Phi(\Gamma)$, any admissible vector field $\Lambda$ for the couple $\left(\Phi(\Gamma), \Phi\left(\overline{\Gamma}\right)\right)$ may be (globally) extended, by using left translations, to an admissible vector field $\tilde{\Lambda}$ for the couple $\left(\Gamma', \Phi\left(\overline{\Gamma}\right)\right)$. In fact, the extension $\tilde{\Lambda}$ is, by construction, a left-invariant vector field on $\Gamma'$ and its flow at the identities is completely contained in $\Phi\left(\overline{\Gamma}\right)$. On the other hand, analogously, the restriction to $\Phi(\Gamma)$ of any admissible vector field $\Theta$ for the couple $\left(\Gamma', \Phi\left(\overline{\Gamma}\right)\right)$ is an admissible vector field for the couple $\left(\Phi(\Gamma), \Phi\left(\overline{\Gamma}\right)\right)$. Therefore, the characteristic distribution of $\overline{\Gamma}$ as a subgroupoid of $\Phi\left(\overline{\Gamma}\right)$ is the restriction of the characteristic distribution of $\overline{\Gamma}$ as a subgroupoid of $\overline{\Gamma}'$. □

Thus, these results show a *consistency property* in the definition of the characteristic distribution. In particular, the characteristic foliation (respectively, distribution) does not depend on the "*ambient space*."

Note that, analogously to Theorem 1, we may prove that the base-characteristic distribution $A\overline{\Gamma}^{\sharp}$ is integrable. Thus, we denote the foliation which integrates the base-characteristic distribution over the base $M$ by $\mathcal{F}$. For each point $x \in M$, the leaf of $\mathcal{F}$ containing $x$ will be denoted by $\mathcal{F}(x)$. Here $\mathcal{F}$ is called the *base-characteristic foliation of* $\overline{\Gamma}$.

**Example 7.** Let $\sim$ be an equivalence relation on a manifold $M$, i.e., a binary relation that is reflexive, symmetric, and transitive. Then, define the subset $\mathcal{O}$ of $M \times M$ given by

$$\mathcal{O} := \{(x, y) \ : \ x \sim y\}. \tag{5}$$

Hence, $\mathcal{O}$ is a subgroupoid of $M \times M$ over $M$. In fact, this is equivalent to the reflexive, symmetric, and transitive properties. For each $x \in M$, we denote by $\mathcal{O}_x$ to the orbit around $x$,

$$\mathcal{O}_x := \{y \ : \ x \sim y\}.$$

Note that the orbits divide $M$ into a disjoint union of subsets. However, these are not (necessarily) submanifolds.

On the other hand, the base-characteristic foliation gives us a foliation $\mathcal{F}$ of $M$ such that

$$\mathcal{F}(x) \subseteq \mathcal{O}_x, \quad \forall x \in M.$$

Thus, consider an arbitrary equivalence relation on a manifold $M$. Perhaps the orbits are not manifolds but we have proved that *we may divide M in a maximal foliation such that any orbit is a union of leaves.* This foliation is maximal in the sense that there is no any other coarser foliation of $M$ whose leaves are contained in the orbits (see Theorem 4 and Corollary 5).

Next, we show that the leaves of $\mathcal{F}$ may be endowed with even more geometric structure. Indeed, we construct a Lie groupoid structure over each leaf of $\mathcal{F}$.

For each $x \in M$, let us consider the groupoid $\overline{\Gamma}(\mathcal{F}(x))$ generated by $\overline{\mathcal{F}}(\epsilon(x))$. Note that, for each $\overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$,

$$\overline{\mathcal{F}}(\epsilon(x)) = \overline{\mathcal{F}}\left(\overline{h}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right).$$

Hence,

$$\overline{\mathcal{F}}\left(\overline{h}^{-1}\right) = \overline{h}^{-1} \cdot \overline{\mathcal{F}}(\epsilon(x)) = \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right).$$

On the other hand, let $\bar{t} \in \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right)$. Therefore,

$$\overline{\mathcal{F}}\left(\overline{h} \cdot \bar{t}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\bar{t}\right) = \overline{h} \cdot \overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right) = \overline{\mathcal{F}}(\epsilon(x)).$$

i.e., $\overline{h} \cdot \bar{t} \in \overline{\mathcal{F}}(\epsilon(x))$ and, hence, $\bar{t}$ can be written as $\overline{h}^{-1} \cdot \overline{g}$ with $\overline{g} \in \overline{\mathcal{F}}(\epsilon(x))$. Thus, we have proved that

$$\overline{\mathcal{F}}\left(\epsilon\left(\alpha\left(\overline{h}\right)\right)\right) \subset \overline{\Gamma}(\mathcal{F}(x)),$$

for all $\overline{h} \in \overline{\mathcal{F}}(\epsilon(x))$. In fact, by following the same argument we have that

$$\overline{\Gamma}(\mathcal{F}(x)) = \sqcup_{\overline{g} \in \overline{\mathcal{F}}(\epsilon(x))} \overline{\mathcal{F}}(\epsilon(\alpha(\overline{g}))), \tag{6}$$

i.e., $\overline{\Gamma}(\mathcal{F}(x))$ can be depicted as a disjoint union of fibers at the identities. Furthermore, $\overline{\Gamma}(\mathcal{F}(x))$ may be equivalently defined as the smallest transitive subgroupoid of $\overline{\Gamma}$ which contains $\overline{\mathcal{F}}(\epsilon(x))$. Observe that the $\beta$-fiber of this groupoid at a point $y \in \mathcal{F}(x)$ is given by $\overline{\mathcal{F}}(\epsilon(y))$. Hence, the $\alpha$-fiber at $y$ is

$$\overline{\mathcal{F}}^{-1}(\epsilon(y)) = i \circ \overline{\mathcal{F}}(\epsilon(y)).$$

Moreover, the groups $\overline{\mathcal{F}}(\epsilon(y)) \cap \Gamma_y$ are exactly the isotropy groups of $\overline{\Gamma}(\mathcal{F}(x))$.

**Theorem 4.** *For each $x \in M$ there exists a transitive Lie subgroupoid $\overline{\Gamma}(\mathcal{F}(x))$ of $\Gamma$ with base $\mathcal{F}(x)$.*

The proof of this result comes from some technical lemmas and may be found in [5, 6].

Thus, we have divided the manifold $M$ into leaves $\mathcal{F}(x)$ which have a maximal structure of transitive Lie subgroupoids of $\Gamma$.

**Corollary 5.** (de León et al. [5]) *Let $\mathcal{H}$ be a foliation of $M$ such that for each $x \in M$ there exists a transitive Lie subgroupoid $\Gamma(x)$ of $\Gamma$ over the leaf $\mathcal{H}(x)$ contained in $\overline{\Gamma}$ whose family of $\beta$-fibers defines a foliation on $\Gamma$. Then, the base-characteristic foliation $\mathcal{F}$ is coarser than $\mathcal{H}$, i.e.,*

$$\mathcal{H}(x) \subseteq \mathcal{F}(x), \quad \forall x \in M.$$

*Furthermore, it satisfies that*

$$\Gamma(x) \subseteq \overline{\Gamma}(\mathcal{F}(x)).$$

As a consequence, we have that $\overline{\Gamma}$ is a transitive Lie subgroupoid of $\Gamma$ if, and only if, $M = \mathcal{F}(x)$ and $\overline{\Gamma} = \overline{\Gamma}(\mathcal{F}(x))$ for some $x \in M$.

Let us consider the following equivalence relation $\sim$ on $M$ given by

$$x \sim y \quad \Leftrightarrow \quad \exists\, \overline{g} \in \overline{\Gamma}, \quad \alpha(\overline{g}) = x, \quad \beta(\overline{g}) = y.$$

Then, by Example 7, we have a subgroupoid $\overline{\Gamma}^B$ of the pair groupoid $M \times M$. Thus, we may consider its associated base-characteristic distribution $A\overline{\Gamma}^B$ at $M$, which is called the *transitive distribution of* $\overline{\Gamma}$. The associated base-characteristic foliation $\mathcal{G}$ of $M$ will be called the *transitive foliation of* $\overline{\Gamma}$.

**Corollary 6.** *The base-characteristic foliation $\mathcal{F}$ based on the groupoid $\overline{\Gamma}$ is contained in the transitive foliation $\mathcal{G}$ of $\overline{\Gamma}$.*

**Proof.** For each $x \in M$, $\mathcal{F}(x) \times \mathcal{F}(x)$ defines a transitive Lie subgroupoid of $M \times M$ over $\mathcal{F}(x)$, and the result follows from Corollary 5. □

Summarizing, for a fixed subgroupoid $\overline{\Gamma}$ of a Lie groupoid $\Gamma$ we have available three canonical foliations, $\overline{\mathcal{F}}$, $\mathcal{F}$, and $\mathcal{G}$. Roughly speaking, $\mathcal{G}$ divides the base manifold into a maximal foliation such that each leaf is transitive or, in other words, $\mathcal{G}$ divides the orbits of $\overline{\Gamma}$ into a maximal foliation of $M$. The main difference between the foliations $\mathcal{G}$ and $\mathcal{F}$ is that, with $\mathcal{F}$, we are requiring "*differentiability*" not only on the base manifold $M$ but also on the groupoid $\overline{\Gamma}$.

For instance, suppose that $\overline{\Gamma}$ is a transitive subgroupoid of $\Gamma$. Then, $\mathcal{G}$ consists of one unique leaf equal to $M$. However, if $\overline{\Gamma}$ is not a Lie subgroupoid of $\Gamma$ the base-characteristic foliation $\mathcal{F}$ does not have (necessarily) one unique leaf equal to $M$.

Apart from Example 7, we may study several relevant applications of the characteristic distribution. In [5] we may find some of them. Here we are mainly interested in the so-called *material distributions*, which are presented in the following.

## 3. Elastic simple materials

We start by dealing with the notion of *simple material*. For a detailed introduction to this topic, we refer to the books [11, 22, 27]. Another recommended reference is [28].

A *(deformable) body* is defined as an oriented manifold $\mathcal{B}$ of dimension three which can be covered by just one chart. The points of the body $\mathcal{B}$ are called *body points* or *material particles* and are denoted by using capital letters ($X, Y, Z \in \mathcal{B}$). A *sub-body* of $\mathcal{B}$ is an open subset $\mathcal{U}$ of the manifold $\mathcal{B}$.

The existence of the so-called *configurations* arises from the need to manifest the body into the "*real world*." Thus, a configuration is an embedding $\phi : \mathcal{B} \to \mathbb{R}^3$. An *infinitesimal configuration at a particle $X$* is given by the 1-jet $j^1_{X,\phi(X)}\phi$ where $\phi$ is a configuration of $\mathcal{B}$. The points on the Euclidean space $\mathbb{R}^3$ are called *spatial points* and are denoted by lowercase letters ($x, y, z \in \mathbb{R}^3$).

From now on, we fix a configuration, denoted by $\phi_0$, called the *reference configuration*. The image $\mathcal{B}_0 = \phi_0(\mathcal{B})$ is called the *reference state*. Coordinates in the reference configuration are denoted by $X^I$, whereas any other coordinates are denoted by $x^i$.

A *deformation* of the body $\mathcal{B}$ is defined as the change of configurations $\kappa = \phi_1 \circ \phi_0^{-1}$ or, equivalently, a diffeomorphism from the reference state $\mathcal{B}_0$ to any other open subset $\mathcal{B}_1$ of $\mathbb{R}^3$. Analogously, an *infinitesimal deformation at $\phi_0(X)$* is given by a 1-jet $j^1_{\phi_0(X),\phi(X)}\kappa$ where $\kappa$ is a deformation.

A relevant goal in continuum mechanics is to study the motion of a body. Here, the internal properties of the body will play an important role (rubber and rock are not deformed equally under the same loading).

We may interpret this statement as the fact that the dynamical principles are not enough to characterize the motion of a deformable body. Thus, following [1, 2], the mechanical response of the body to the history of its deformations is supposed to be determined by the so-called *constitutive equations*.

For *elastic simple bodies* [27] we assume that the constitutive law depends at each particle only on the instantaneous infinitesimal deformation at the same particle. More explicitly, the *mechanical response* for an (elastic) simple material $\mathcal{B}$, in a fixed reference configuration $\phi_0$, is formalized as a differentiable map $W$ from the set $\mathcal{B} \times Gl(3, \mathbb{R})$, where $Gl(3, \mathbb{R})$ is the general linear group of $3 \times 3$-regular matrices, to a fixed (finite-dimensional) vector space $V$. In general, $V$ is the space of *stress tensors*.

Indeed, in continuum mechanics, the contact forces at a particle $X$ in a given configuration $\phi$ are characterized by a symmetric second-order tensor

$$T_{X,\phi} : \mathbb{R}^3 \to \mathbb{R}^3$$

on $\mathbb{R}^3$ called the *stress tensor*. A physical interpretation is that $T_{X,\phi}$ assigns to the unit normal of a smooth surface a stress vector acting on the surface at $\phi(X)$. Then, the mechanical response is given by the following equation:

$$W(X,F) = T_{X,\phi},$$

where $F$ is the 1-jet at $\phi_0(X)$ of $\phi \circ \phi_0^{-1}$.

At this point, we should introduce the *rule of change of reference configuration*. In particular, let $\phi_1$ be another configuration and $W_1$ be the mechanical response associated with $\phi_1$. Then,

$$W_1(X,F) = W(X, F \cdot C_{01}), \tag{7}$$

for all regular matrices $F$ where $C_{01}$ is the matrix associated with the 1-jet at $\phi_0(X)$ of $\phi_1 \circ \phi_0^{-1}$. Equivalently,

$$W(X,F_0) = W_1(X,F_1), \tag{8}$$

where $F_i$, $i = 0, 1$, is the matrix associated with the 1-jet at $\phi_i(X)$ of $\phi \circ \phi_i^{-1}$ with $\phi$ a configuration. It is important to remark that Equation (7) implies that we may define $W$ as a map on the space of 1-jets of (local) configurations which is *independent on the chosen reference configuration*. In fact, for each configuration $\phi$ we define

$$W\left(j^1_{X,x}\phi\right) = W(X,F),$$

where $F$ is the matrix associated with the 1-jet at $\phi_0(X)$ of $\phi \circ \phi_0^{-1}$.

Note that any sub-body inherits the structure of an elastic simple body from the body $\mathcal{B}$. This local property permits us to compare the material properties at different particles of the body. In particular, we may investigate when two particles $X$ and $Y$ are made of the same material. To do this, we introduce the notion of *material isomorphisms*.

**Definition 8.** Let $\mathcal{B}$ be a body. Two material particles $X, Y \in \mathcal{B}$ are said to be *materially isomorphic* if there exists a local diffeomorphism $\psi$ from an open neighborhood $\mathcal{U} \subseteq \mathcal{B}$ of $X$ to an open neighborhood $\mathcal{V} \subseteq \mathcal{B}$ of $Y$ such that $\psi(X) = Y$ and

$$W(X, F \cdot P) = W(Y,F), \tag{9}$$

for all infinitesimal deformations $F$ where $P$ is given by the Jacobian matrix of $\phi_0 \circ \psi \circ \phi_0^{-1}$ at $\phi_0(X)$. The 1-jets of local diffeomorphisms satisfying Equation (9) are called *material isomorphisms*. A material isomorphism from $X$ to itself is called a *material symmetry*. In cases where it causes no confusion we often refer to the associated matrix $P$ the material isomorphism (or symmetry).

Intuitively, two points are materially isomorphic if the constitutive equation of one of them differs from the other only by an application of a linear transformation, i.e., they are made of the same material.

The relation of being "*materially isomorphic*" defines an equivalence relation (symmetric, reflexive, and transitive) over the body manifold $\mathcal{B}$.

We denote by $G(X)$ the set of all material symmetries at particle $X$. As a consequence, we have that every $G(X)$ is a group. Therefore, we may prove that the material symmetry groups of materially isomorphic particles are conjugated, i.e., if $X$ and $Y$ are material isomorphic we have that

$$G(Y) = P \cdot G(X) \cdot P^{-1},$$

where $P$ is a material isomorphism from $X$ to $Y$.

**Proposition 7.** *Let $\mathcal{B}$ be a body. Two body points $X$ and $Y$ are materially isomorphic if, and only if, there exist two (local) configurations $\phi_1$ and $\phi_2$ such that*

$$W_1(X,F) = W_2(Y,F), \quad \forall F,$$

*where $W_i$ is the mechanical response associated with $\phi_i$ for $i = 1, 2$.*

**Proof.** Let $j^1_{X,Y}\psi$ be a material isomorphism from $X$ to $Y$. Then, we may prove the result by imposing,

$$\phi_2 = \phi_1 \circ \psi,$$

where $\phi_i$ is the reference configuration for $W_i$. □

This result is crucial to understand the idea of material isomorphism. Thus, two material points will be made of the same material if their mechanical responses are the same up to a change of reference configuration.

**Definition 9.** A body $\mathcal{B}$ is said to be *uniform* if all of its body points are materially isomorphic.

Intuitively, a body is uniform if all the points are made of the same material. Let $\mathcal{B}$ be a uniform body and a fixed body point $X_0$; for any other body point $Y$ we may find a material isomorphism from $X_0$ to $Y$, say $P(Y) \in Gl(3,\mathbb{R})$. Then, we construct a map $P : \mathcal{B} \to Gl(3,\mathbb{R})$ consisting of material isomorphisms. However, $P$ is not in general a differentiable map.

**Definition 10.** A body $\mathcal{B}$ is said to be *smoothly uniform* if for each point $X \in \mathcal{B}$ there is a neighborhood $\mathcal{U}$ around $X$ and a smooth map $P : \mathcal{U} \to Gl(3,\mathbb{R})$ such that for all $Y \in \mathcal{U}$ it satisfies that $P(Y)$ is a material isomorphism from $X$ to $Y$. The map $P$ is called a *right (local) smooth field of material isomorphisms*. A *left (local) smooth field of material isomorphisms* is defined analogously.

Let $P$ be a right (local) smooth field of material isomorphisms. Hence, the mechanical response of the sub-body $\mathcal{U}$ satisfies that

$$W(Y,F) = W(X, F \cdot P(Y)),$$

for all $Y \in \mathcal{U}$. Then, we may define

$$\overline{W}(F) = W(X,F).$$

Therefore,

$$W(Y,F) = \overline{W}(F \cdot P(Y)). \tag{10}$$

Equation (10) means that the dependence of the mechanical response (near a material particle) of the body is given by a multiplication of $F$ to the right by a right smooth field of material isomorphisms.

**Proposition 8.** *Let $\mathcal{B}$ be a body. Then, $\mathcal{B}$ is (smoothly) uniform if, and only if, there exists a (differentiable) map $\overline{W} : Gl(3,\mathbb{R}) \to V$ satisfying Equation (10) for a (differentiable) map $P : \mathcal{U} \to Gl(3,\mathbb{R})$.*

**Proof.** Assume that Equation (10) is satisfied for a map $P$ and fix a material point $X$. Then, consider

$$Q : \mathcal{U} \to Gl(3,\mathbb{R})$$

given by

$$Q(Y) = P(Y)P(X)^{-1}. \tag{11}$$

Therefore, $Q$ is a left (smooth) field of material isomorphisms. □

It is important to note that smooth uniformity is the starting point of the use of $G$-structures in [3] (see [29, 30]; see also [31, 32]). In fact, let us consider a smoothly uniform body $\mathcal{B}$. Fix $Z_0 \in \mathcal{B}$ and $\overline{Z}_0 = j^1_{0,Z_0}\phi \in F\mathcal{B}$ a frame at $Z_0$. Then, the following set:

$$\omega_{G_0}(\mathcal{B}) := \{j^1_{Z_0,Y}\psi \cdot \overline{Z}_0, \, : \, j^1_{Z_0,Y}\psi \text{ is a material isomorphism}\},$$

is a $G_0$-structure on $\mathcal{B}$ (which contains $\overline{Z}_0$). This $G_0$-structure has been used to study simple materials. However, it is defined only for smoothly uniform bodies, and it is not canonically defined.

The use of groupoids solved these two points as may be found in [33, 34] (see also [4, 35]). Let $\mathcal{B}$ be an elastic simple body with reference configuration $\phi_0$, and mechanical response $W : \mathcal{B} \times Gl(3,\mathbb{R}) \to V$. Equation (8) permits us to define $W$ on the space of (local) configurations in such a way that for any configuration $\phi$ we have that

$$W\left(j^1_{X,x}\phi\right) = W(X,F),$$

where $F$ is the matrix associated with the 1-jet at $\phi_0(X)$ of $\phi \circ \phi_0^{-1}$. Indeed, composing $\phi_0$ by the left, we obtain that $W$ may be described as a differentiable map $W : \Pi^1(\mathcal{B}, \mathcal{B}) \to V$ from the groupoid of 1-jets $\Pi^1(\mathcal{B}, \mathcal{B})$ (see Example 3) to the vector space $V$ which does not depend on the image point of the 1-jets of $\Pi^1(\mathcal{B}, \mathcal{B})$, i.e., for all $X, Y, Z \in \mathcal{B}$.

$$W\left(j^1_{X,Y}\phi\right) = W\left(j^1_{X,Z}\left(\phi_0^{-1} \circ \tau_{Z-Y} \circ \phi_0 \circ \phi\right)\right), \tag{12}$$

for all $j^1_{X,Y}\phi \in \Pi^1(\mathcal{B}, \mathcal{B})$, where $\tau_v$ is the translation map on $\mathbb{R}^3$ by the vector $v$. It is relevant to note here that, in contrast with the definition on the space of a local configuration, the definition of the mechanical response on $\Pi^1(\mathcal{B}, \mathcal{B})$ does not depend on the choice of the configuration $\phi_0$.

Therefore, we can say that two material particles $X$ and $Y$ are materially isomorphic if, and only if, there exists a local diffeomorphism $\psi$ from an open subset $\mathcal{U} \subseteq \mathcal{B}$ of $X$ to an open subset $\mathcal{V} \subseteq \mathcal{B}$ of $Y$ such that $\psi(X) = Y$ and

$$W\left(j^1_{Y,\kappa(Y)}\kappa \cdot j^1_{X,Y}\psi\right) = W\left(j^1_{Y,\kappa(Y)}\kappa\right), \tag{13}$$

for all $j^1_{Y,\kappa(Y)}\kappa \in \Pi^1(\mathcal{B}, \mathcal{B})$. Under these conditions, $j^1_{X,Y}\psi$ will be called a material *isomorphism from $X$ to $Y$*.

For any two points $X, Y \in \mathcal{B}$, the collection of all material isomorphisms from $X$ to $Y$ will be denoted by $G(X, Y)$. Then, the set

$$\Omega(\mathcal{B}) = \cup_{X,Y\in\mathcal{B}} G(X, Y) \tag{14}$$

is a subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. This groupoid will be called the *material groupoid of $\mathcal{B}$*.

The material symmetry group $G(X)$ at a body point $X \in \mathcal{B}$ is simply the isotropy group of $\Omega(\mathcal{B})$ at $X$. For any $X \in \mathcal{B}$, the set of material isomorphisms from $X$ to any other point (respectively, from any point to $X$) will be denoted by $\Omega(\mathcal{B})_X$ (respectively, $\Omega(\mathcal{B})^X$). Finally, the structure maps of $\Omega(\mathcal{B})$ will be denoted by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\epsilon}$, and $\overline{i}$ which are just the restrictions of the corresponding ones on $\Pi^1(\mathcal{B}, \mathcal{B})$.

As a consequence of the continuity of $W$ we have that, for all $X \in \mathcal{B}$, $G(X)$ is a closed subgroup of $\Pi^1(\mathcal{B}, \mathcal{B})^X_X$ and, therefore, we have the following result.

**Proposition 9.** *Let $\mathcal{B}$ be a simple body. Then, for all $X \in \mathcal{B}$, the symmetry group $G(X)$ is a Lie subgroup of $\Pi^1(\mathcal{B}, \mathcal{B})^X_X$.*

This result might convey the impression that $\Omega(\mathcal{B})$ is a Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$. However, this is not true (see [3, 13] for some counterexamples).

**Proposition 10.** *Let $\mathcal{B}$ be a body. Here $\mathcal{B}$ is uniform if and only if $\Omega(\mathcal{B})$ is a transitive subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.*

Next, by composing appropriately with the reference configuration, smooth uniformity (Definition 10) may be characterized in the following way.

**Proposition 11.** *A body $\mathcal{B}$ is smoothly uniform if, and only if, for each point $X \in \mathcal{B}$ there is a neighborhood $\mathcal{U}$ around $X$ such that for all $Y \in \mathcal{U}$ and $j^1_{Y,X}\phi \in \Omega(\mathcal{B})$ there exists a local section $\mathcal{P}$ of*

$$\overline{\alpha}_X : \Omega(\mathcal{B})^X \to \mathcal{B},$$

*from $\epsilon(X)$ to $j^1_{Y,X}\phi$.*

For obvious reasons, (local) sections of $\overline{\alpha}_X$ will be called *left fields of material isomorphism at $X$*. On the other hand, local sections of

$$\overline{\beta}^X : \Omega(\mathcal{B})_X \to \mathcal{B},$$

will be called *right fields of material isomorphism at $X$*.

Thus, $\mathcal{B}$ is smoothly uniform if, and only if, for any two particles $X, Y \in \mathcal{B}$ there are two open neighborhoods $\mathcal{U}, \mathcal{V} \subseteq \mathcal{B}$ around $X$ and $Y$, respectively, and $\mathcal{P} : \mathcal{U} \times \mathcal{V} \to \Omega(\mathcal{B}) \subseteq \Pi^1(\mathcal{B}, \mathcal{B})$, a differentiable section of the anchor map $\left(\overline{\alpha}, \overline{\beta}\right)$. When $X = Y$, we may assume $\mathcal{U} = \mathcal{V}$ and $\mathcal{P}$ is a morphism of groupoids over the identity map, i.e.,

$$\mathcal{P}(Z, T) = \mathcal{P}(R, T)\,\mathcal{P}(Z, R), \ \forall T, R, Z \in \mathcal{U}.$$

Thus, we have the following corollary of Proposition 9.

**Corollary 12.** *Let $\mathcal{B}$ be a body. Here $\mathcal{B}$ is smoothly uniform if and only if $\Omega(\mathcal{B})$ is a transitive Lie subgroupoid of $\Pi^1(\mathcal{B}, \mathcal{B})$.*

**Proof.** Assume that $\mathcal{B}$ is smoothly uniform. Consider $j^1_{X,Y}\psi \in \Omega(\mathcal{B})$ and $\mathcal{P} : \mathcal{U} \times \mathcal{V} \to \Omega(\mathcal{B})$, a differentiable section of the anchor map $\left(\overline{\alpha}, \overline{\beta}\right)$ with $X \in \mathcal{U}$ and $Y \in \mathcal{V}$. Then, we may construct the following bijection

$$\begin{array}{rccl} \Psi_{\mathcal{U},\mathcal{V}} : & \Omega(\mathcal{U},\mathcal{V}) & \to & \mathcal{B}\times\mathcal{B}\times G(X,Y) \\ & j^1_{Z,T}\phi & \mapsto & \left(Z, T, \mathcal{P}(Z,Y)\left[j^1_{Z,T}\phi\right]^{-1}\mathcal{P}(X,T)\right) \end{array}$$

where $\Omega(\mathcal{U},\mathcal{V})$ is the set of material isomorphisms from $\mathcal{U}$ to $\mathcal{V}$. By using Proposition 9, we deduce that $G(X,Y)$ is a differentiable manifold. Thus, we can endow $\Omega(\mathcal{B})$ with a differentiable structure of a manifold. Finally, the converse has been proved in [11]). □

This corollary is useful to understand the difference between smooth uniformity and ordinary uniformity. Furthermore, it provides an intuition about the lack of differentiability which the material groupoid could have. Thus arises the need of using the characteristic distribution (see the previous section).

Consider $\mathcal{B}$ as a simple body with $W : \Pi^1(\mathcal{B},\mathcal{B}) \to V$ as the mechanical response. Then, we have available the so-called material groupoid $\Omega(\mathcal{B})$ which is a (not necessarily Lie) subgroupoid of the groupoid of 1-jets $\Pi^1(\mathcal{B},\mathcal{B})$. It makes sense to apply here the notion of characteristic distribution.

Let $\Theta$ be an admissible vector field for the couple $\left(\Pi^1(\mathcal{B},\mathcal{B}), \Omega(\mathcal{B})\right)$, i.e., its local flow at the identity $\epsilon(X)$, $\varphi_t^{\Theta}(\epsilon(X))$, satisfies that

$$\varphi_t^{\Theta}(\epsilon(X)) \subseteq \Omega(\mathcal{B})$$

for all $X \in \mathcal{B}$ and $t$ in the domain of the flow at $\epsilon(X)$. Therefore, for any $g \in \Pi^1(\mathcal{B},\mathcal{B})$, we have

$$\begin{aligned} TW(\Theta(g)) &= \frac{\partial}{\partial t_{|0}}\left(W\left(\varphi_t^{\Theta}(g)\right)\right) \\ &= \frac{\partial}{\partial t_{|0}}\left(W\left(g\cdot\varphi_t^{\Theta}(\epsilon(\alpha(g)))\right)\right) \\ &= \frac{\partial}{\partial t_{|0}}(W(g)) = 0. \end{aligned}$$

Hence, we obtain

$$TW(\Theta) = 0 \tag{15}$$

The converse is proved in a similar way.

The characteristic distribution $A\Omega(\mathcal{B})^T$ of the material groupoid is called the *material distribution*. It is generated by the (left-invariant) vector fields on $\Pi^1(\mathcal{B},\mathcal{B})$ which are in the kernel of $TW$. The base-characteristic distribution $A\Omega(\mathcal{B})^\sharp$ (see Theorem 1) is called the *body-material distribution*, and the transitive distribution is called the *uniform-material distribution*.

The foliations associated with the material distribution, the body-material distribution, and the uniform-material distribution are called *material foliation*, *body-material foliation*, and *uniform-material foliation*, and they are denoted by $\overline{\mathcal{F}}$, $\mathcal{F}$ and $\mathcal{G}$, respectively.

For each $X \in \mathcal{B}$, we denote the Lie groupoid $\Omega(\mathcal{B})(\mathcal{F}(X))$ by $\Omega(\mathcal{F}(X))$ (see Theorem 4). Denote the groupoid of all material isomorphisms at points in $\mathcal{G}(X)$ by $\Omega(\mathcal{G}(X))$. Recall that $\Omega(\mathcal{F}(X))$ is a subgroupoid of $\Omega(\mathcal{G}(X))$, i.e., $\Omega(\mathcal{F}(X)) \leq \Omega(\mathcal{G}(X))$. In fact, in the general case, *the condition of maximality on the leaves of $\mathcal{G}$ means that $\mathcal{G}$ is the coarsest foliation such that, at each leaf $\mathcal{G}(X)$, the groupoid of all material isomorphisms at points in $\mathcal{G}(X)$ is a transitive subgroupoid of $\overline{\Gamma}$.*

Observe that in continuum mechanics a *sub-body* of a body $\mathcal{B}$ is given by an open submanifold of $\mathcal{B}$. Here, however, the foliation $\mathcal{F}$ gives us submanifolds of different dimensions (not only dimension three). Thus, we follow [3, 13] for a more general definition.

**Definition 11.** A *material submanifold (or generalized sub-body) of $\mathcal{B}$* is a submanifold of $\mathcal{B}$.

It is important to note that any generalized sub-body $\mathcal{P}$ inherits certain material structure from $\mathcal{B}$. In particular, the material response of a material submanifold $\mathcal{P}$ is measured by restricting $W$ to the 1-jets of local diffeomorphisms $\phi$ on $\mathcal{B}$ from $\mathcal{P}$ to $\mathcal{P}$. However, it is easy to observe that a material submanifold of a body is not exactly a body. See [5] for a discussion on this subject.

As a corollary of Theorem 1 and Corollary 5, we have the following result.

**Theorem 13.** *The body-material foliation $\mathcal{F}$ (respectively, uniform material foliation $\mathcal{G}$) divides the body $\mathcal{B}$ into maximal smoothly uniform material submanifolds (respectively, uniform material submanifolds).*

It should be observed that, in this case, "maximal" means that any other foliation $\mathcal{H}$ by smoothly uniform material submanifolds (respectively, uniform material submanifolds) is thinner than $\mathcal{F}$ (respectively, $\mathcal{G}$), i.e.,

$$\mathcal{H}(X) \subseteq \mathcal{F}(X) \text{ (respectively, } \mathcal{G}(X)), \ \forall X \in \mathcal{B}.$$

Therefore, the application of material distributions has been used to prove this rather intuitive result: *Let $\mathcal{B}$ be a general (smoothly uniform or not) simple material. Then, $\mathcal{B}$ may be decomposed into "(smoothly) uniform parts" and this decomposition is, in fact, a foliation of the material body.*

The material distributions are useful to define new notions of *graded uniformity* and *generalized homogeneity* (see [13]). However, here we are interested in another application of the characteristic distributions. In particular, we want to study the notion of *material evolution*.

## 4. Material evolution

We now present a formulation of the time evolution of a material body, following mainly [6–8]. In our geometrical description of the theory of simple bodies, the time variable has not played a role. Our body is, in some sense, frozen. Nevertheless, in many practical applications, the material properties of the body may change with time. A relevant example is given by the volumetric growth and remodeling of biological tissues, such as bone and muscle.

Thus, material evolution is, roughly speaking, the temporal counterpart of the notion of material body. In the case of a material body, we compare the constitutive response of two different material particles at the same instant of time. On the other hand, in material evolution we study the constitutive properties of different points at different instants of time.

We, therefore, consider a *body-time manifold* as the fiber bundle $\mathcal{C} = \mathbb{R} \times \mathcal{B}$ over $\mathbb{R}$. For the sake of simplicity, time and space are supposed to be absolute, but may be extended to a more general case.

**Definition 12.** A *history* is given by a fiber bundle embedding $\Phi : \mathcal{C} \to \mathbb{R} \times \mathbb{R}^3$ over the identity.

Equivalently, $\Phi$ can be seen as a differentiable family of configurations $\phi_t : \mathcal{B} \to \mathbb{R}^3$ such that

$$\phi(t, X) = \phi_t(X) = (pr_2 \circ \Phi)(t, X), \quad \forall t \in \mathbb{R}, \ \forall X \in \mathcal{B}, \tag{16}$$

where $pr_2 : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the projection on the second component.

The map $\Phi$, therefore, represents the evolution of the body in such a way that the configuration of $\mathcal{B}$ at time $t$ is $\phi_t$. Then, at each instant of time $t$, one may consider the infinitesimal configuration as the 1-jet $j^1_{X,\phi_t(X)}\phi_t$.

Next, we need to introduce the constitutive law of the material evolution. In the framework of simple bodies, we assume that, for a fixed reference configuration $\phi_0$, the constitutive response at each material particle $X$ and at each instant of time $t$ may be characterized by one or more functions that depend on the matrices $F$ associated with the infinitesimal configurations $j^1_{X,\phi_t(X)}\phi_t$ at particle $X$ and time $t$. Accordingly, the *mechanical response* will be a differentiable map

$$W : \mathcal{C} \times Gl(3, \mathbb{R}) \to V,$$

where $V$ is again a real vector space (generally, $V$ will be assumed to be the space of stress tensors). The definition of the mechanical response permits us to compare material responses at different particles at different instants of time.

Once again, the construction of the mechanical response seems to be restricted to a fixed reference configuration. To clarify this dependence we have the *rule of change of reference configuration*.

Thus, consider a different configuration $\phi_1$, and $W_1$ its associated mechanical response. We establish that

$$W_1(t, X, F) = W(t, X, F \cdot C_{01}), \tag{17}$$

for all regular matrices $F$, where $C_{01}$ is the matrix associated with the 1-jet at $\phi_0(X)$ of $\phi_1 \circ \phi_0^{-1}$. Equivalently,

$$W(t, X, F_0) = W_1(t, X, F_1), \tag{18}$$

where $F_i$, $i = 0, 1$, is the matrix associated with the 1-jet at $\phi_i(X)$ of $\phi \circ \phi_i^{-1}$ with $\phi$ a configuration.

Therefore, Equation (17) allows us to define $W$ over the space of (local) histories which is *independent on the chosen reference configuration*. In fact, for each history $\Phi = \phi_t$ we define

$$W(t, X, \Phi) = W\left(j^1_{X,x}\phi_t\right) = W(t, X, F_t), \tag{19}$$

where $F_t$ is the matrix associated with the 1-jet $j^1_{\phi_0(X),x}\left(\phi_t \circ \phi_0^{-1}\right)$ at $\phi_0(X)$. Reciprocally, for each point $(t, X)$ any differentiable map $W_{(t,X)}$ on the space of (local) histories defines a constitutive functional at $(t, X)$ by Equation (19) satisfying the rule of change of reference configuration (17).

Observe that, for all $t$, the manifold $\{t\} \times \mathcal{B}$ inherits the structure of a simple body by restricting the mechanical response $W$ to the history of deformations at the same instant $t$. This body is called *state at t of the body* $\mathcal{B}$ and is denoted by $\mathcal{B}_t$. As long as it invites no confusion, we refer to the simple body $\{0\} \times \mathcal{B}$ as the material body $\mathcal{B}$.

On the other hand, it is also important to state that the mechanical response defines a structure of material evolution on any sub-body $\mathcal{U}$ of the body $\mathcal{B}$ by restriction. Nevertheless, analogously to Definition 11, we need to relax the definition of "*material evolution*" to permit variation of material submanifolds with time.

**Definition 13.** A *material evolution* for a *material submanifold* (or *body-time generalized sub-body*) of $\mathcal{C}$ is a submanifold $\mathcal{M}$ of $\mathcal{C}$.

Thus, for each instant $t$ (such that there exists a particle $X$ with $(t, X) \in \mathcal{M}$) we have that the *state at t of the material submanifold* is

$$(\{t\} \times \mathcal{B}) \cap \mathcal{M} = \{t\} \times \mathcal{M}_t,$$

for a submanifold $\mathcal{M}_t$ of $\mathcal{B}$. Hence, varying $t$ we may see how the material submanifold $\mathcal{M}_t$ changes over time. Note that, we do no impose that

$$\mathcal{M} = I \times \mathcal{N},$$

for an interval $I$ and a material submanifold $\mathcal{N}$ because we are permitting variations in the "*shape*" of $\mathcal{N}$.

**Definition 14.** Let $\mathcal{C}$ be a body-time manifold. Two pairs $(t, X), (s, Y) \in \mathcal{C}$ are said to be *materially isomorphic* if there exists a local diffeomorphism $\psi$ from an open neighborhood $\mathcal{U} \subseteq \mathcal{B}$ of $X$ to an open neighborhood $\mathcal{V} \subseteq \mathcal{B}$ of $Y$ such that $\psi(X) = Y$ and

$$W(t, X, F \cdot P) = W(s, Y, F), \tag{20}$$

for all infinitesimal deformation $F$ where $P$ is given by the Jacobian matrix of $\phi_0 \circ \psi \circ \phi_0^{-1}$ at $\phi_0(X)$. The 1-jets of local diffeomorphisms satisfying Equation (20) are called *time-material isomorphisms* (or *material isomorphisms* if there is no danger of confusion) from $(t, X)$ to $(s, Y)$. A material isomorphism from $(t, X)$ to itself is called a *time-material symmetry* or *material symmetry*.

Roughly speaking, two pairs $(t, X), (s, Y) \in \mathcal{C}$ are materially isomorphic if the material points $X$ and $Y$ are made of the same material at the instants $t$ and $s$, respectively. As a particular case, we may consider that $(t, X)$ and $(s, X)$ are materially isomorphic for all $t$ and $s$ in an interval $I$. Then, the constitutive properties of the material particle $X$ do not change in the time interval $I$.

We denote by $G(t, X)$ the set of all material symmetries at $(t, X)$. Again, any $G(t, X)$ is a group.

**Proposition 14.** *Let $\mathcal{C}$ be a body-time manifold. Two body pairs $(t, X)$ and $(s, Y)$ are materially isomorphic if, and only if, there exist two (local) configurations $\phi_1$ and $\phi_2$ such that*

$$W_1(t, X, F) = W_2(s, Y, F), \quad \forall F,$$

*where $W_i$ is the mechanical response associated with $\phi_i$ for $i = 1, 2$.*

**Proof.** Consider $j^1_{X,Y}\psi$ a material isomorphism from $(t, X)$ to $(s, Y)$. We choose $\phi_1 = \phi_0$ and

$$\phi_2 = \phi_1 \circ \psi$$

Then,

$$\begin{aligned} W_1(t,X,F) &= W_1(s,Y,F\cdot P)\\ &= W_2(s,Y,F\cdot P\cdot C_{21})\\ &= W_2(s,Y,F), \end{aligned}$$

where $P$ is the Jacobian matrix of $\phi_1\circ\psi\circ\phi_1^{-1}$ at $\phi_1(X)$. □

By analogy with the time-independent case, this result reflects the intuitive idea that two points are materially isomorphic if their constitutive properties are the same.

### 4.1. Material evolution groupoids

Let us consider the vertical subbundle associated with the body-time manifold $\mathcal{C}$, $\mathcal{V}$, and the associated frame groupoid (see Example 3) $\Phi(\mathcal{V})\rightrightarrows\mathcal{C}$. Note that, for all $(t,X)\in\mathcal{C}$, we have that

$$\mathcal{V}_{(t,X)} = \{0\}\times T_X\mathcal{B}.$$

At this point, it is important to highlight that the groupoid $\Phi(\mathcal{V})\rightrightarrows\mathcal{C}$ will be relevant in what follows. In fact, *the role of* $\Phi(\mathcal{V})\rightrightarrows\mathcal{C}$ *for material evolution is comparable with the role of the* 1*-jets groupoid* $\Pi^1(\mathcal{B},\mathcal{B})$ *on* $\mathcal{B}$ *for elastic simple materials* (see Section 3).

The reader could now raise a natural question: *why do we take this groupoid instead of* $\Pi^1(\mathcal{C},\mathcal{C})$ *or any other of its subgroupoids?*

The answer is simple: the elements of this groupoid may be identified with the 1-jets of the histories (see Definition 12) via a reference configuration.

Let $\Phi:\mathcal{C}\to\mathcal{C}$ be a (local) embedding of fiber bundles over the identity on $\mathbb{R}$. Then, an element of $\Phi(\mathcal{V})$ may be given by a triple $\left((t,X),\Phi(t,X),j^1_{X,\phi_t(X)}\phi_t\right)$, where

$$\phi(t,X) = \phi_t(X) = (pr_2\circ\Phi)(t,X),\quad \forall t\in\mathbb{R},\ \forall X\in\mathcal{B}.$$

Another, less-intuitive but easier way to represent an element of $\Phi(\mathcal{V})$, is a triple $\left(t,s,j^1_{X,Y}\phi\right)$ with $s,t\in\mathbb{R}$, $X\in\mathcal{B}$ and $\phi$ a local automorphism on $\mathcal{B}$ from $X$ to $Y$. Then, the local coordinates of $\Phi(\mathcal{V})$ (see Equation (6)) are given by

$$\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{W}}\right):\left(t,s,x^i,y^j,y^j_i\right), \tag{21}$$

where, for each $\left(t,s,j^1_{X,Y}\phi\right)\in\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{W}}\right)$:

- $t\left(t,s,j^1_{X,Y}\phi\right) = t$;
- $s\left(t,s,j^1_{X,Y}\phi\right) = s$;
- $x^i\left(t,s,j^1_{X,Y}\phi\right) = x^i(X)$;
- $y^j\left(t,s,j^1_{X,Y}\phi\right) = y^j(Y)$;
- $y^j_i\left(t,s,j^1_{X,Y}\phi\right) = \dfrac{\partial\left(y^j\circ\phi\right)}{\partial x^i_{|X}}$;

where $\left(x^i\right)$ and $\left(y^i\right)$ are local charts defined on the open subsets of $\mathcal{B},\mathcal{U}$, and $\mathcal{W}$, respectively, and $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{W}}\right)$ is given by the triples $\left(t,s,j^1_{X,Y}\phi\right)$ such that $X\in\mathcal{U}$ and $Y\in\mathcal{W}$.

Note that, the space of (local) embeddings $\Phi:\mathcal{C}\to\mathcal{C}$ of fiber bundles over the identity on $\mathbb{R}$ is easily identified with the set of (local) histories by using the reference configuration. Thus, *the groupoid* $\Phi(\mathcal{V})\rightrightarrows\mathcal{C}$ *encompasses all the possible histories of the material evolution*. Then, by using Equation (19), we may define $W$ on the space $\Phi(\mathcal{V})$,

$$W:\Phi(\mathcal{V})\to V,$$

as follows,

$$W\left(t,s,j^1_{X,Y}\phi\right) = W(t,X,\Phi),$$

such that

$$\Phi(s,Y) = (s, \phi_0 \circ \phi(Y)), \quad \forall (s,Y) \in \mathcal{C},$$

where $\phi_0$ is the reference configuration. Then, $W$ does not depend on the final point, i.e., for all $(t,X),(s,Y),(r,Z) \in \mathcal{C}$

$$W\left(t,s,j^1_{X,Y}\phi\right) = W\left(t,r,j^1_{X,Z}\left(\phi_0^{-1} \circ \tau_{Z-Y} \circ \phi_0 \circ \phi\right)\right), \tag{22}$$

for all $\left(t,s,j^1_{X,Y}\phi\right) \in \Phi(\mathcal{V})$ where $\tau_v$ is the translation map on $\mathbb{R}^3$ by the vector $v$. This point of view is useful for our purpose.

**Definition 15.** The *material groupoid* of a body-time manifold with mechanical response $W$ is defined as the largest subgroupoid $\Omega(\mathcal{C}) \rightrightarrows \mathcal{C}$ of $\Phi(\mathcal{V})$ such that leaves $W$ invariant. More explicitly, an element of $\Phi(\mathcal{V})$ $\left(t,s,j^1_{X,Y}\phi\right)$ is in the material groupoid if and only if

$$W\left(t,r,j^1_{X,Z}(\psi \cdot \phi)\right) = W\left(s,r,j^1_{Y,Z}\psi\right),$$

for all $\left(s,r,j^1_{Y,Z}\psi\right) \in \Phi(\mathcal{V})$.

In other words, $\Omega(\mathcal{C})$ is the space of all (time-)material isomorphisms (see Definition 14). This groupoid was first presented in [7].

The isotropy group at each $(t,X) \in \mathcal{C}$ will be denoted by $G(t,X)$, and its elements are the material symmetries at $(t,X)$. Observe that, as in the purely spatial case, the resulting groupoid does not have to be a Lie subgroupoid of $\Phi(\mathcal{V}) \rightrightarrows \mathcal{C}$.

**Definition 16.** We define the $(t,s)$-*material groupoid* $\Omega_{t,s}(\mathcal{B})$ as the set of all material isomorphisms from the instant $t$ to the instant $s$.

Note that, when $t = s$, the $(t,t)$-material groupoid $\Omega_{t,t}(\mathcal{B})$ is a subgroupoid of the material groupoid $\Omega(\mathcal{C})$. For each instant $t$, $\Omega_{t,t}(\mathcal{B})$ is called *t-material groupoid* and denoted by $\Omega_t(\mathcal{B})$.

On the other hand, $\Omega_t(\mathcal{B})$ may be considered as a subgroupoid of $\Pi^1(\mathcal{B},\mathcal{B})$, where we are identifying $\mathcal{B}$ with $\{t\} \times \mathcal{B}$. Note that, indeed, $\Omega_t(\mathcal{B})$ *is the material groupoid associated with the simple-body structure of the state t of the body* $\mathcal{B}$*, i.e., with this identification,*

$$\Omega_t(\mathcal{B}) = \Omega(\mathcal{B}_t).$$

We use both interpretations of $\Omega_t(\mathcal{B})$ indistinctly throughout the paper.

As a transversal construction, we define the $(X,Y)$-*material groupoid* $\Omega_{X,Y}(\mathbb{R})$.

**Definition 17.** The $(X,Y)$-*material groupoid* $\Omega_{X,Y}(\mathbb{R})$ is defined as the set of all material isomorphisms from the particle $X$ to the particle $Y$ varying the time variable.

Again, we may note that, when $X = Y$, the $(X,X)$-material groupoid $\Omega_{X,X}(\mathbb{R})$ is a subgroupoid of the material groupoid $\Omega(\mathcal{C})$. For each material point $X$, $\Omega_{X,X}(\mathbb{R})$ is called the $X$*-material groupoid* and denoted by $\Omega_X(\mathbb{R})$.

On the other hand, $\Omega_X(\mathbb{R})$ may be considered as a subgroupoid of $(\mathbb{R}\times\mathbb{R}) \times \Pi^1(\mathcal{B},\mathcal{B})^X_X \rightrightarrows \mathbb{R}$, where we are identifying $\mathbb{R}$ with $\mathbb{R} \times \{X\}$. Furthermore, the structure of Lie groupoid of $(\mathbb{R}\times\mathbb{R}) \times \Pi^1(\mathcal{B},\mathcal{B})^X_X$ is given by

$$\left(s,t,j^1_{X,X}\phi\right) \cdot \left(r,s,j^1_{X,X}\psi\right) = \left(r,t,j^1_{X,X}(\phi \circ \psi)\right),$$

for all $\left(s,t,j^1_{X,X}\phi\right), \left(r,s,j^1_{X,X}\psi\right) \in \mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$. Again, we use both interpretations of $\Omega_X(\mathbb{R})$ throughout the paper.

Roughly speaking, the material groupoid $\Omega(\mathcal{C}) \rightrightarrows \mathcal{C}$ encompasses the global evolution of the body, the $t$-material groupoid $\Omega_t(\mathcal{B})$ encodes all the material properties of the body at the instant $t$, and the $X$-material groupoid $\Omega_X(\mathbb{R})$ embraces all the evolution of the particle $X$.

**Proposition 15.** *Let* $\Omega(\mathcal{C})$ *be the material groupoid. If* $\Omega(\mathcal{C})$ *is a Lie subgroupoid of* $\Phi(\mathcal{V})$*, then for all instants t and all material points X we have that* $\Omega_t(\mathcal{C})$ *and* $\Omega_X(\mathbb{R})$ *are Lie subgroupoids of* $\Phi(\mathcal{V})$*.*

**Proof.** Assume that $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$. Let us consider the following submersions,

$$\pi_1 : \Omega(\mathcal{C}) \to \mathcal{B} \times \mathcal{B}, \quad \pi_2 : \Omega(\mathcal{C}) \to \mathbb{R} \times \mathbb{R},$$

given by

$$\pi_1\left(t,s,j^1_{X,Y}\phi\right) = (X,Y), \quad \pi_2\left(t,s,j^1_{X,Y}\phi\right) = (t,s),$$

for all $\left(t,s,j^1_{X,Y}\phi\right) \in \Omega(\mathcal{C})$. Then

$$\Omega_X(\mathbb{R}) = \pi_1^{-1}(X,X), \quad \Omega_t(\mathcal{C}) = \pi_2^{-1}(t,t).$$

□

Thus, the condition of "*being a Lie groupoid*" is stronger over the material groupoid than over the $t$-material groupoids and $X$-material groupoids.

Note that, all the defined canonical groupoids satisfy the following short sequences of contents,

$$\Omega_t(\mathcal{B}) \leq \Omega(\mathcal{C}) \leq \Phi(\mathcal{V}), \quad \forall t,$$

$$\Omega_X(\mathbb{R}) \leq \Omega(\mathcal{C}) \leq \Phi(\mathcal{V}), \quad \forall X.$$

Then, we may construct the corresponding characteristic distributions. We start with the characteristic distribution $A\Omega(\mathcal{C})^T$ associated with the material groupoid, which is called the *material distribution* of the body-time manifold $\mathcal{C}$. In a similar way to the construction of the material distribution associated with a *spatial* body $\mathcal{B}$ (see Equation (15)), $A\Omega(\mathcal{C})^T$ is generated by the (left-invariant) vector fields on $\Phi(\mathcal{V})$ which are in the kernel of $TW$. Equivalently, the material distribution of $\mathcal{C}$ is generated by the left-invariant vector fields $\Theta$ on $\Phi(\mathcal{V})$ such that

$$TW(\Theta) = 0. \tag{23}$$

Let $\Theta$ be a left-invariant vector field on $\Phi(\mathcal{V})$. Then,

$$\Theta\left(t,s,x^i,y^j,y^i_j\right) = \lambda\frac{\partial}{\partial t} + \Theta^i\frac{\partial}{\partial x^i} + y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j} \tag{24}$$

with respect to a local system of coordinates $\left(t,s,x^i,y^j,y^i_j\right)$ on $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{V}}\right)$ with $\mathcal{U}$ and $\mathcal{V}$ two open subsets of $\mathcal{B}$ and $\Phi\left(\mathcal{V}_{\mathcal{U},\mathcal{V}}\right)$, is given by the triples $\left(t,s,j^1_{X,Y}\phi\right)$ in $\Phi(\mathcal{V})$ such that $X \in \mathcal{U}$ and $Y \in \mathcal{V}$. Then, $\Theta$ is an admissible vector field for the couple $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$ if, and only if, the following equation holds

$$\lambda\frac{\partial W}{\partial t} + \Theta^i\frac{\partial W}{\partial x^i} + y^i_l\Theta^l_j\frac{\partial W}{\partial y^i_j} = 0. \tag{25}$$

Note that, here $\lambda$, $\Theta^i$, and $\Theta^i_j$ are functions depending on $t$ and $X$. Thus, the construction of the material distribution is reduced to solving Equation (25). The base-characteristic distribution $A\Omega(\mathcal{C})^\sharp$ is called the *body-material distribution*, and the transitive distribution $A\Omega(\mathcal{B})^B$ is called the *uniform-material distribution*.

The foliations associated with the material distribution, the body-material distribution, and the uniform-material distribution are called *material foliation*, *body-material foliation*, and *uniform-material foliation*, and are denoted by $\overline{\mathcal{F}}$, $\mathcal{F}$, and $\mathcal{G}$, respectively.

On the other hand, analogously, for an instant $t$, consider the $t$-material groupoid $\Omega_t(\mathcal{B})$ as the material groupoid of the state at $t$ of the material body $\mathcal{B}$. Therefore (see Equation (15)), the characteristic distribution $A\Omega_t(\mathcal{B})^T$ associated with the $t$-material groupoid, which is called the *t-material distribution*, is generated by the (left-invariant) vector fields on $\Pi^1(\mathcal{B},\mathcal{B})$ which are in the kernel of $TW_t$, where $W_t$ is given by

$$W_t : \Pi^1(\mathcal{B},\mathcal{B}) \to V,$$

such that $W_t\left(j^1_{X,Y}\phi\right) = W\left(t,t,j^1_{X,Y}\phi\right)$ for all $j^1_{X,Y}\phi \in \Pi^1(\mathcal{B},\mathcal{B})$. In other words, the $t$-material distribution of $\mathcal{C}$ is generated by the left-invariant vector fields $\Theta$ on $\Pi^1(\mathcal{B},\mathcal{B})$ such that

$$TW_t(\Theta) = 0. \tag{26}$$

Indeed, it satisfies the condition that $A\Omega_t(\mathcal{B})^T$ is the material distribution of the state at $t$ of the material body.

Let $\Theta$ be a left-invariant vector field on $\Pi^1(\mathcal{B},\mathcal{B})$. Then,

$$\Theta\left(x^i,y^j,y^i_j\right)=\Theta^i\frac{\partial}{\partial x^i}+y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j}, \tag{27}$$

with respect to a local system of coordinates $\left(x^i,y^j,y^i_j\right)$ on $\Pi^1(\mathcal{U},\mathcal{U})$ (see Equation (3)) with $\mathcal{U}$ an open subset of $\mathcal{B}$. Then, $\Theta$ is an admissible vector field for the couple $\left(\Pi^1(\mathcal{B},\mathcal{B}),\Omega_t(\mathcal{B})\right)$ if, and only if, the following equation holds

$$\Theta^i\frac{\partial W_t}{\partial x^i}+y^i_l\Theta^l_j\frac{\partial W_t}{\partial y^i_j}=0. \tag{28}$$

Note that, here, $\Theta^i$ and $\Theta^i_j$ are functions depending on $X$.

On the other hand, let us observe that, taking into account the *consistency proposition* (Proposition 3), as a subgroupoid of $\Phi(\mathcal{V})$, the groupoid $\Omega_t(\mathcal{B})$ is generated by the left-invariant vector fields $\Theta$ on $\Phi(\mathcal{V})$,

$$\Theta\left(t,s,x^i,y^j,y^i_j\right)=\lambda\frac{\partial}{\partial t}+\Theta^i\frac{\partial}{\partial x^i}+y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j} \tag{29}$$

such that $\lambda_{|\{t\}\times\{t\}\times\Pi^1(\mathcal{B},\mathcal{B})}\equiv 0$ and

$$\Theta^i\frac{\partial W_t}{\partial x^i}+y^i_l\Theta^l_j\frac{\partial W_t}{\partial y^i_j}=0 \tag{30}$$

on any material point at the instant $t$. The base-characteristic distribution $A\Omega_t(\mathcal{B})^\sharp$ (see Theorem 1) is called the *t-body-material distribution* and the transitive distribution $A\Omega_t(\mathcal{B})^B$ (see Corollary 6) is called the *t-uniform-material distribution*.

The foliations associated with the $t$-material distribution, the $t$-body-material distribution, and the $t$-uniform-material distribution are called *t-material foliation*, *t-body-material foliation*, and *t-uniform-material foliation*, and are denoted by $\overline{\mathcal{F}}_t$, $\mathcal{F}_t$, and $\mathcal{G}_t$, respectively.

The characteristic distribution associated with the $X$-material groupoid $A\Omega_X(\mathbb{R})^T$ is called the *X-material distribution*. Analogously, $A\Omega_X(\mathbb{R})^T$ is generated by the (left-invariant) vector fields on $(\mathbb{R}\times\mathbb{R})\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ which are in the kernel of $TW_X$, where $W_X$ is given by the restriction of $W$ to $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$,

$$W_X:\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X\to V.$$

In other words, the $X$-material distribution of $\mathcal{C}$ is generated by the left-invariant vector fields $\Theta$ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ such that

$$TW_X(\Theta)=0. \tag{31}$$

Note that the groupoid structure of $(\mathbb{R}\times\mathbb{R})\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$ is the unique groupoid structure such that it is a subgroupoid of $\Phi(\mathcal{V})$, i.e.,

$$\left(s,t,j^1_{X,X}\phi\right)\cdot\left(r,s,j^1_{X,X}\psi\right)=\left(r,t,j^1_{X,X}(\phi\circ\psi)\right),$$

for all $\left(s,t,j^1_{X,X}\phi\right),\left(r,s,j^1_{X,X}\psi\right)\in\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$.

Let $\Theta$ be a left-invariant vector field on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{B},\mathcal{B})^X_X$. Then,

$$\Theta\left(t,s,y^i_j\right)=\lambda\frac{\partial}{\partial t}+y^i_l\Theta^l_j\frac{\partial}{\partial y^i_j}, \tag{32}$$

with respect to a local system of coordinates $\left(t,s,y^i_j\right)$ on $\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{U},\mathcal{U})^X_X$ with $\mathcal{U}$ an open subset of $\mathcal{B}$ with $X\in\mathcal{U}$. Then, $\Theta$ is an admissible vector field for the couple $(\Phi(\mathcal{V}),\Omega_X(\mathbb{R}))$ if, and only if, the following equation holds,

$$\lambda\frac{\partial W_X}{\partial t}+y^i_l\Theta^l_j\frac{\partial W_X}{\partial y^i_j}=0. \tag{33}$$

Observe that, here $\lambda$ and $\Theta^i_j$ are functions depending on $t$.

On the other hand, taking into account the *consistency proposition* (Proposition 3), as a subgroupoid of $\Phi(\mathcal{V})$, the groupoid $\Omega_X(\mathbb{R})$ is generated by the left-invariant vector fields $\Theta$ on $\Phi(\mathcal{V})$,

$$\Theta\left(t, s, x^i, y^j, y^i_j\right) = \lambda \frac{\partial}{\partial t} + \Theta^i \frac{\partial}{\partial x^i} + y^i_l \Theta^l_j \frac{\partial}{\partial y^i_j} \tag{34}$$

such that $\Theta^i_{|\mathbb{R}\times\mathbb{R}\times\Pi^1(\mathcal{U},\mathcal{U})^X_X} \equiv 0$ and

$$\Theta^i \frac{\partial W_X}{\partial x^i} + y^i_l \Theta^l_j \frac{\partial W_X}{\partial y^i_j} = 0, \tag{35}$$

at any instant for a fixed material point $X$. The base-characteristic distribution $A\Omega_X(\mathbb{R})^\sharp$ (see Theorem 1) is called *$X$-body-material distribution* and the transitive distribution $A\Omega_X(\mathbb{R})^B$ (see Corollary 6) is called *$X$-uniform-material distribution*.

The foliations associated with the $X$-material distribution, the $X$-body-material distribution, and the $X$-uniform-material distribution are called *$X$-material foliation*, *$X$-body-material foliation*, and *$X$-uniform-material foliation*, and are denoted by $\overline{\mathcal{F}}_X$, $\mathcal{F}_X$, and $\mathcal{G}_X$, respectively. It is important not to confuse $\overline{\mathcal{F}}_X$ (respectively, $\mathcal{F}_X$ and $\mathcal{G}_X$), the $X$-material foliation (respectively, $X$-body-material foliation and $X$-uniform-material foliation), with $\overline{\mathcal{F}}(\epsilon(X))$ (respectively, $\mathcal{F}(X)$ and $\mathcal{G}(X)$), the leaf at $\epsilon(X)$ (respectively, the leaf at $X$) of the foliation $\overline{\mathcal{F}}$ (respectively, $\mathcal{F}$ and $\mathcal{G}$).

To summarize, regarding a body-time manifold $\mathcal{C}$, we have constructed the following canonical short sequences of groupoids

$$\Omega_t(\mathcal{B}) \leq \Omega(\mathcal{C}) \leq \Phi(\mathcal{V}), \quad \forall t,$$
$$\Omega_X(\mathbb{R}) \leq \Omega(\mathcal{C}) \leq \Phi(\mathcal{V}), \quad \forall t,$$

and the following canonical short sequences of distributions:

$$A\Omega_t(\mathcal{B})^T \leq A\Omega(\mathcal{C})^T \leq T\Phi(\mathcal{V}), \quad \forall t,$$
$$A\Omega_X(\mathbb{R})^T \leq A\Omega(\mathcal{C})^T \leq T\Phi(\mathcal{V}), \quad \forall t,$$
$$A\Omega_t(\mathcal{B})^B \leq A\Omega(\mathcal{C})^B \leq T\mathcal{C}, \quad \forall t,$$
$$A\Omega_X(\mathbb{R})^B \leq A\Omega(\mathcal{C})^B \leq T\mathcal{C}, \quad \forall t,$$
$$A\Omega_t(\mathcal{B})^\sharp \leq A\Omega(\mathcal{C})^\sharp \leq T\mathcal{C}, \quad \forall t,$$
$$A\Omega_X(\mathbb{R})^\sharp \leq A\Omega(\mathcal{C})^\sharp \leq T\mathcal{C}, \quad \forall t.$$

## 5. Remodeling

As opposed to the uniformity in the spatial case, new material properties arise associated with the evolution of the body. In particular, the temporal counterpart of uniformity is a specific case of evolution of the material called *remodeling*. This section is focused on the study of *global remodeling*, as one of the main contributions of this paper.

**Definition 18.** Let $\mathcal{C}$ be a body-time manifold.

- A material particle $X \in \mathcal{B}$ undergoes a process of *remodeling* when its time evolution is such that its responses at all instants are related by a material isomorphism, i.e., all the points of $\mathbb{R} \times \{X\}$ are connected materially isomorphically.
- Here $\mathcal{C}$ represents a *global remodeling*, or simply a *remodeling*, when all the material points undergo a remodeling.
- We say that $\mathcal{C}$ represents a *uniform remodeling* when it represents a remodeling and one (and, hence, every) state is uniform.
- *Growth* and *resorption* are given by a remodeling with mass increase or decrease of the material body $\mathcal{B}$.

Intuitively, a material evolution is a remodeling when the intrinsic constitutive properties of the material do not change with time. This kind of evolution may be found in biological tissues [9]. Wolff's law of the trabecular architecture of bones (see, for instance, [10]) is a relevant example. Here, the trabeculae are assumed to change their orientation following the principal direction of stress. It is important to note that the fact that the material body remains materially isomorphic over time does not preclude the possibility of adding (growth) or removing (resorption) material, as long as the material added is *of the same type* as the substrate.

**Proposition 16.** *Let $\mathcal{C}$ be a body-time manifold. A material particle $X \in \mathcal{B}$ undergoes remodeling if, and only if, the $X$-material groupoid $\Omega_X(\mathbb{R})$ is transitive. Here $\mathcal{C}$ represents a remodeling if, and only if, for all material points $X$, the $X$-material groupoid $\Omega_X(\mathbb{R})$ is transitive.*

**Corollary 17.** *Let $\mathcal{C}$ be a body-time manifold. The material groupoid $\Omega(\mathcal{C})$ is transitive if, and only if, $\mathcal{C}$ represents a uniform remodeling.*

Observe that, analogously to uniformity, the definition of remodeling is pointwise. Consider a material particle $X_0$ which undergoes a remodeling or, equivalently, such that there exists a map

$$P : \mathbb{R} \to Gl(3, \mathbb{R}) \tag{36}$$

such that, for all $t \in \mathbb{R}$, $P(t)$ is a material isomorphism from $(t_0, X_0)$ to $(t, X_0)$ for a fixed time $t_0$. Nevertheless, the differentiability condition of $P$ is not guaranteed.

**Definition 19.** Let $\mathcal{C}$ be a body-time manifold. A material point $X_0$ is said to undergo a *smooth remodeling* if for each point $t \in \mathbb{R}$ there is an interval $I$ around $t$ and a smooth map $P : I \to Gl(3, \mathbb{R})$ such that for all $s \in I$ it satisfies that $P(s)$ is a material isomorphism from $(t, X_0)$ to $(s, X_0)$. The map $P$ is called a *right (local) smooth remodeling process at $X_0$*. A *left (local) smooth remodeling process at $X_0$* is defined in a similar way.

### 5.1. Mass consistency condition

Note that the definition of material isomorphism does not include any relation to the mass density of the body. However, it is desirable to impose some kind of condition to be consistent with the mass density.

Thus, for each instant of time, a volume form is specified, i.e., we have $\omega(t)$ a time-dependent volume form on $\mathcal{B}$. Let $X_0$ be a material particle undergoing remodeling. Without loss of generality, we assume that the remodeling process $P$ satisfies the initial condition,

$$P(0) = I.$$

Then, *mass consistency condition* [6, 8] consists of the imposition on the remodeling process at $X_0$ that it must preserve the volume form. In other words, a (local) right smooth remodeling process $P$ at $X_0$ satisfies the mass consistency condition if, and only if,

$$P(t)^* \omega(t) = \omega(0), \quad \forall t \in I.$$

Then, equivalently, the associated mass density, $\rho(t) = |\omega(t)|$, should satisfy that

$$\rho(t) = |J_{P(t)}|^{-1} \rho(0), \tag{37}$$

where $J_{P(t)}$ is the determinant of $P(t)$. We also assume that $P$ is orientation-preserving, i.e., $J_{P(t)} > 0$.

Calculating the time derivatives of Equation (37), we obtain

$$\begin{aligned}
\dot{\rho}(t) &= \left(\dot{J^{-1}_{P(t)}} \rho(0)\right) \\
&= -\rho(0) J^{-2}_{P(t)} \left[\dot{J}_{P(t)}\right] \\
&= -\rho(0) J^{-2}_{P(t)} \left[J_{P(t)} Tr\left(P^{-1}(t) \cdot \dot{P}(t)\right)\right] \\
&= -\rho(0) J^{-1}_{P(t)} Tr\left(P^{-1}(t) \cdot \dot{P}(t)\right) \\
&= -\rho(t)\, Tr\left(P^{-1}(t) \cdot \dot{P}(t)\right).
\end{aligned}$$

The term $L_{P(t)} = P^{-1}(t) \cdot \dot{P}(t)$ is called the *remodeling velocity gradient*.

**Proposition 18.** *Let $\mathcal{C}$ be a body-time manifold and $X_0$ be a material particle. A remodeling process $P$ is producing growth if, and only if, the trace of the remodeling velocity gradient is negative. Conversely, resorption is equivalent to a positive trace of the remodeling velocity gradient.*

**Proof.** The trace of the remodeling velocity gradient $L_{P(t)} = P^{-1}(t) \cdot \dot{P}(t)$ is negative (respectively, positive) if, and only if, $\rho$ is an increasing (respectively, decreasing) function or, in other words, the volume of $\mathcal{B}$ respect to $\omega(t)$ is increasing (respectively, decreasing). □

Several interesting examples of remodeling processes may be found in the literature. In particular, in [8] a model for orthotropic solids is proposed in which the tensor $P$ is proper orthogonal at all times. This model simulates an evolution law in trabecular bones.

Let us assume that $\mathcal{B}$ is uniform. Then, $\mathcal{B}$ is uniform in all its states. Thus, for a fixed point $(t_0, X_0) \in \mathcal{C}$ we may find a map

$$P : \mathcal{C} \to Gl(3, \mathbb{R}) \tag{38}$$

such that, for all $(t, Y) \in \mathcal{C}$, $P(t, Y)$ is a material isomorphism from $(t_0, X_0)$ to $(t, Y)$. *However, even when all the particles undergo smooth remodeling, $P$ does not have to be differentiable*. In other words, roughly speaking, the evolution of all the particles over time could be "*smooth*," but the change from the time-evolution of one particle to another could still be "*abrupt*" (not differentiable).

Thus, because we cannot define *smooth remodeling* over the whole material evolution as the smooth remodeling at all points, we still need a more restrictive definition of smoothness on the evolution of the material body.

**Definition 20.** A body-time manifold $\mathcal{C}$ is said to represent a *smooth uniform remodeling* if for each point $(t, X) \in \mathcal{C}$ there is a neighborhood $\mathcal{U}$ around $(t, X)$ and a smooth map $P : \mathcal{U} \to Gl(3, \mathbb{R})$ such that for all $(s, Y) \in \mathcal{U}$ it satisfies that $P(s, Y)$ is a material isomorphism from $(t, X)$ to $(s, Y)$. The map $P$ is called a *right (local) smooth field of material isomorphisms*. A *left (local) smooth field of material isomorphisms* is defined analogously.

One may think that it is reasonable to expect that a non-uniform body can undergo a smooth remodeling. However, the definition of this (more general) kind of smooth remodeling is not clear. One of the contributions of this paper is the use of material distributions to define and characterize this kind of smooth remodeling for non-uniform bodies (Definition 21).

**Proposition 19.** *Let $\mathcal{C}$ be a body-time manifold. Then, $\mathcal{C}$ represents a (smooth) uniform remodeling if, and only if, there exist (differentiable) maps $\overline{W} : Gl(3, \mathbb{R}) \to V$ and $P : \mathcal{U} \to Gl(3, \mathbb{R})$ covering $\mathcal{C}$ satisfying*

$$W(s, Y, F) = \overline{W}(F \cdot P(s, Y)). \tag{39}$$

**Proof.** The proof of this proposition is analogous to that of Proposition 8. □

Let us consider now $W$ as a map on $\Phi(\mathcal{V})$.

**Proposition 20.** *Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a smooth uniform remodeling if, and only if, for each instant $t$ and each material point $X$ there is an open neighborhood $\mathcal{D} \subset \mathcal{C}$ around $(t, X)$ such that for all $(s, Y) \in \mathcal{D}$ and $\left(s, t, j^1_{Y,X}\phi\right) \in \Omega(\mathcal{C})$ there exists a local section $\mathcal{P}$ of the source map $\overline{\alpha}$ of $\Omega(\mathcal{C})$ to the $\overline{\beta}$-fiber $\Omega(\mathcal{C})^{(t,X)}$,*

$$\overline{\alpha}_{(t,X)} : \Omega(\mathcal{C})^{(t,X)} \to \mathcal{C},$$

*from $\epsilon(t, X)$ to $\left(s, t, j^1_{Y,X}\phi\right)$.*

For these reasons, (local) sections of $\overline{\alpha}_{(t,X)}$ are called *left local (smooth) field of material isomorphisms at* $(t, X)$. On the other hand, local sections of

$$\overline{\beta}^{(t,X)} : \Omega(\mathcal{C})_{(t,X)} \to \mathcal{C}$$

are called *right local (smooth) fields of material isomorphisms at* $(t, X)$.

Hence, $\mathcal{C}$ represents a smooth uniform remodeling if, and only if, for any points $(t, X)$ and $(s, Y)$, there are two open neighborhoods $\mathcal{D}$ and $\mathcal{E}$, respectively, and a differentiable map

$$\mathcal{P} : \mathcal{D} \times \mathcal{E} \to \Omega(\mathcal{C}) \subseteq \Phi(\mathcal{V}),$$

which is a section of the anchor map $(\alpha, \beta)$ of $\Phi(\mathcal{V})$. When $t = s$ we may assume $\mathcal{D} = \mathcal{E}$ and $\mathcal{P}$ is a morphism of groupoids over the identity map, i.e.,

$$\mathcal{P}((r, Z), (l, T)) = \mathcal{P}((m, S), (l, T))\, \mathcal{P}((r, Z), (m, S)),$$

for all $(r, Z), (l, T), (m, S) \in \mathcal{D}$. These kinds of maps are called *local (smooth) fields of material isomorphisms.*

**Corollary 21.** *Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a smooth uniform remodeling if, and only if, $\Omega(\mathcal{C})$ is a transitive Lie subgroupoid of $\Phi(\mathcal{V})$.*

**Proof.** Suppose that $\mathcal{C}$ represents a smooth uniform remodeling. Let there be a triple $\left(s, t, j^1_{Y,X}\phi\right) \in \Omega(\mathcal{C})$ and a local (smooth) field of material isomorphisms through $\left(s, t, j^1_{Y,X}\phi\right)$,

$$\mathcal{P} : \mathcal{D} \times \mathcal{E} \to \Omega(\mathcal{C}) \subseteq \Phi(\mathcal{V}) \subseteq \Phi(\mathcal{V}).$$

Then, the local structure of manifold is given by the charts $\Psi_{\mathcal{D},\mathcal{E}} : \Omega(\mathcal{D}, \mathcal{E}) \to \mathbb{R} \times \mathbb{R} \times \Omega(\mathcal{C})^{(t,X)}_{(s,Y)}$ such that

$$\Psi_{\mathcal{D},\mathcal{E}}\left(k, l, j^1_{Z,T}\psi\right) = \left(k, l, \mathcal{P}((l, T), (t, X))\left[\left(k, l, j^1_{Z,T}\psi\right)\right]\mathcal{P}((s, Y), (k, Z))\right),$$

for all $\left(k, l, j^1_{Z,T}\psi\right) \in \Omega(\mathcal{D}, \mathcal{E})$. Here, $\Omega(\mathcal{D}, \mathcal{E})$ is the set of material isomorphisms from $\mathcal{D}$ to instants at $\mathcal{E}$. □

Again, we have here a clear difference between a process of remodeling of a uniform body and a process of *smooth* remodeling of a uniform body (see Corollary 18).

Of course, the existence of fields of material isomorphisms is not canonical. Indeed, for a (local) smooth field of material isomorphisms

$$\mathcal{P} : \mathcal{D} \times \mathcal{D} \to \Omega(\mathcal{C}) \subseteq \Phi(\mathcal{V}),$$

any other remodeling process $\mathcal{Q}$ satisfies that

$$\mathcal{Q}((s, Y), (k, Z)) \in \mathcal{P}((t_0, X_0), (k, Z)) \cdot \Omega(\mathcal{C})^{(t_0,X_0)}_{(t_0,X_0)} \cdot \mathcal{P}((s, Y), (t_0, X_0)),$$

for a fixed point $(t_0, X_0)$ at $\mathcal{D}$. Thus, the symmetry groups of $\Omega(\mathcal{C})$ act as a measure of the degree of freedom available in the choice of the fields of material isomorphisms.

Let $\Omega(\mathcal{C})$ be the material groupoid associated with the body-time manifold $\mathcal{C}$. Then, we may consider the material distribution $A\Omega(\mathcal{C})$, the body-material distribution $A\Omega(\mathcal{C})^\sharp$, and the uniform-material distribution $A\Omega(\mathcal{C})^B$, and their associated foliations, the material foliation $\overline{\mathcal{F}}$, body-material foliation $\mathcal{F}$, and uniform-material foliation $\mathcal{G}$, respectively.

**Theorem 22.** *Let $\mathcal{C}$ be a body-time manifold. The body-material foliation $\mathcal{F}$ (respectively, uniform material foliation $\mathcal{G}$) divides $\mathcal{C}$ into maximal smooth uniform remodeling processes (respectively, uniform remodeling processes).*

Note that the foliations $\mathcal{F}$ and $\mathcal{G}$ are foliations of the material evolution $\mathcal{C}$. Hence, each leaf is a submanifold of $\mathcal{C}$, i.e., it defines a material evolution of a material submanifold of $\mathcal{B}$ (see Definition 13). Thus, in general, it cannot be properly written as a product space

$$\mathbb{R} \times \mathcal{N}, \tag{40}$$

with $\mathcal{N}$ a submanifold of $\mathcal{B}$. Nevertheless, this impossibility turns out to be the most natural (see Definition 13).

Note that the dimensions of the leaves of the body-material foliation $\mathcal{F}$ (respectively, uniform material foliation $\mathcal{G}$) are the dimensions of the fibers of $A\Omega(\mathcal{C})^\sharp_{(t,X)}$ (respectively, $A\Omega(\mathcal{C})^B_{(t,X)}$). We may, therefore, prove the following result.

**Theorem 23.** *Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a smooth uniform remodeling process (respectively, uniform remodeling) if, and only if,* $\dim\left(A\Omega\left(\mathcal{C}\right)^{\sharp}_{(t,X)}\right) = 4$ *(respectively,* $\dim\left(A\Omega\left(\mathcal{C}\right)^{\mathcal{B}}_{(t,X)}\right) = 4$*) for all instants $t$ and particles $X$, with $A\Omega\left(\mathcal{C}\right)^{\sharp}_{(t,X)}$ (respectively, $A\Omega\left(\mathcal{C}\right)^{\mathcal{B}}_{(t,X)}$) the fiber of $A\Omega\left(\mathcal{C}\right)^{\sharp}$ (respectively, $A\Omega\left(\mathcal{C}\right)^{\mathcal{B}}$) at $(t,X)$.*

This theorem affords us a computational condition for testing the property of being a "*process of remodeling*." In particular, we have to study Equation (25),

$$\lambda\frac{\partial W}{\partial t} + \Theta^{i}\frac{\partial W}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial W}{\partial y^{i}_{j}} = 0, \tag{41}$$

where $\lambda$, $\Theta^{i}$, and $\Theta^{i}_{j}$ are functions depending on $t$ and $X$. The material evolution is a process of remodeling if we may find four linearly independent solutions to this equation.

For each instant $t$, let us recall the $t$-material distribution $A\Omega_{t}\left(\mathcal{B}\right)^{T}$, its associated $t$-body-material distribution $A\Omega_{t}\left(\mathcal{B}\right)^{\sharp}$, and $t$-uniform-material $A\Omega_{t}\left(\mathcal{B}\right)^{\mathcal{B}}$, and the associated foliations $t$-material foliation $\overline{\mathcal{F}}_{t}$, $t$-body-material foliation $\mathcal{F}_{t}$, and $t$-uniform-material foliation $\mathcal{G}_{t}$.

We have proved that the $t$-material groupoid $\Omega_{t}\left(\mathcal{B}\right)$ is just the material groupoid of the state $t$ of the body $\mathcal{B}$. Therefore, by using Theorem 13 we have the following result.

**Theorem 24.** *The $t$-body-material foliation $\mathcal{F}_{t}$ (respectively, $t$-uniform material foliation $\mathcal{G}_{t}$) divides the state $t$ of the body $\mathcal{B}$ into maximal smoothly uniform material submanifolds (respectively, uniform material submanifolds).*

Thus, at any instant of time $t$, we have the body divided into "*smoothly uniform parts*" and we can see how these parts change over time, by letting $t$ vary.

**Proposition 25.** *Let $\mathcal{C}$ be a material evolution and $(t,X)$ be a point in $\mathcal{C}$. Then,*

$$\left(\{t\}\times\mathcal{B}\right)\cap\mathcal{F}\left(t,X\right) = \{t\}\times\mathcal{F}_{t}\left(X\right). \tag{42}$$

**Proof.** Note that, by construction, we have that

$$\{t\}\times\mathcal{F}_{t}\left(X\right)\subseteq\left(\{t\}\times\mathcal{B}\right)\cap\mathcal{F}\left(t,X\right).$$

On the other hand, let $\Theta$ be an admissible vector field for the couple $\left(\Phi\left(\mathcal{V}\right),\Omega\left(\mathcal{C}\right)\right)$. Then, $\Theta$ should satisfy Equation (25), i.e.,

$$\lambda\frac{\partial W}{\partial t} + \Theta^{i}\frac{\partial W}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial W}{\partial y^{i}_{j}} = 0, \tag{43}$$

where

$$\Theta\left(t,s,x^{i},y^{j},y^{i}_{j}\right) = \lambda\frac{\partial}{\partial t} + \Theta^{i}\frac{\partial}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial}{\partial y^{i}_{j}} \tag{44}$$

with respect to a local system of coordinates $\left(t,s,x^{i},y^{j},y^{i}_{j}\right)$ on $\Phi\left(\mathcal{V}_{\mathcal{U}}\right)$ with $\mathcal{U}$ an open subset of $\mathcal{B}$ and $\mathcal{V}_{\mathcal{U}}$ given by the triples $\left(t,s,j^{1}_{X,Y}\phi\right)$ in $\Phi\left(\mathcal{V}\right)$ such that $X,Y\in\mathcal{U}$. Let us consider two cases: □

- We have $T_{(t,X)}\rho\left(\Theta^{\sharp}\left(t,X\right)\right) = 0$, for all projections $\Theta^{\sharp}$ of an admissible vector field $\Theta$ for the couple $\left(\Phi\left(\mathcal{V}\right),\Omega\left(\mathcal{C}\right)\right)$.
  In this case, any admissible vector field $\Theta$ for the couple $\left(\Phi\left(\mathcal{V}\right),\Omega\left(\mathcal{C}\right)\right)$ satisfies that $\lambda\left(t,X\right) = 0$ is the local expression (44). Hence, it satisfies the equation

$$\Theta^{i}\frac{\partial W}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial W}{\partial y^{i}_{j}} = 0. \tag{45}$$

  Therefore, by Equation (28), $\Theta$ is an admissible vector field $\Theta$ for the couple $\left(\Phi\left(\mathcal{V}\right),\Omega_{t}\left(\mathcal{B}\right)\right)$, i.e.,

$$\left(\{t\}\times\mathcal{B}\right)\cap\mathcal{F}\left(t,X\right) = \mathcal{F}\left(t,X\right)\subseteq\{t\}\times\mathcal{F}_{t}\left(X\right).$$

- We have $T_{(t,X)}\rho\left(\Theta^{\sharp}(t,X)\right) \neq 0$, for some projection $\Theta^{\sharp}$ of an admissible vector field $\Theta$ for the couple $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$.
  Then, $T_{(t,X)}\rho\left(A\Omega(\mathcal{C})^{\sharp}\right) = \mathbb{R}$. Thus, we have that

$$T_{(t,X)}(\{t\} \times \mathcal{B}) + T_{(t,X)}\mathcal{F}(t,X) = T_{(t,X)}\mathcal{C},$$

  i.e., $(\{t\} \times \mathcal{B})$ and $\mathcal{F}(t,X)$ are transversal submanifolds of $\mathcal{C}$. Therefore, $(\{t\} \times \mathcal{B}) \cap \mathcal{F}(t,X)$ is a submanifold of $\mathcal{C}$ and

$$T_{(t,X)}\left[(\{t\} \times \mathcal{B}) \cap \mathcal{F}(t,X)\right] = T_{(t,X)}(\{t\} \times \mathcal{B}) \cap T_{(t,X)}\mathcal{F}(t,X).$$

  Thus, the tangent vector fields to $(\{t\} \times \mathcal{B}) \cap \mathcal{F}(t,X)$ are the projections $\Theta^{\sharp}$ of admissible vector fields $\Theta$ for the couple $(\Phi(\mathcal{V}), \Omega(\mathcal{C}))$ such that $\Theta^{\sharp}$ projected on $\mathbb{R}$ is zero, i.e.,

$$\Theta^{i}\frac{\partial W}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial W}{\partial y^{i}_{j}} = 0, \tag{46}$$

  where

$$\Theta\left(t, s, x^{i}, y^{j}, y^{i}_{j}\right) = \Theta^{i}\frac{\partial}{\partial x^{i}} + y^{i}_{l}\Theta^{l}_{j}\frac{\partial}{\partial y^{i}_{j}}. \tag{47}$$

  Then, by Equation (28), $\Theta$ is an admissible vector field $\Theta$ for the couple $(\Phi(\mathcal{V}), \Omega_{t}(\mathcal{B}))$.

In other words, in case we freeze an instant of time $s$ in $\mathcal{F}(t,X)$, we recover the leaf $\{s\} \times \mathcal{F}_{s}(X)$. Thus, if we could write $\mathcal{F}(t,X)$ as in Equation (40), we would be precluding the case in which the shapes of leaves $F_{s}(X)$ change with time, i.e., *each of the leaves $\mathcal{F}(t,X)$ undergoes a remodeling in which the uniform leaves can change*.

Therefore, if the foliations $\mathcal{F}_{t}$ (respectively, $\mathcal{G}_{t}$) permit us to watch how the smoothly uniform leaves (respectively, uniform leaves) of the body change with time, the foliation $\mathcal{F}$ (respectively, $\mathcal{G}$) also shows us how time is divided optimally in such a way that at each interval the material evolution is a smooth remodeling (respectively, remodeling) process of all the leaves at the same time.

Finally, we present a definition of (non-uniform) smooth remodeling inspired by Corollary 21.

**Definition 21.** Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a *smooth remodeling* if $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$ and, for all particles $X$, the $X$-material groupoid $\Omega_{X}(\mathbb{R})$ is a transitive Lie subgroupoid of $\Phi(\mathcal{V})$

Note that, taking into account Proposition 15, if $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$, then for all particles $X$, the $X$-material groupoid $\Omega_{X}(\mathbb{R})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$. Thus, the only requirement on the $X$-material groupoids is transitivity.

Definition 21 expresses mathematically the idea that the material body varies smoothly through time and the intrinsic properties do not change. It is easy to check that the material points undergo a smooth remodeling. In fact, $\Omega_{X}(\mathbb{R})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$ if, and only if, $\mathbb{R}$ can be covered by local sections of the anchor of $\Omega_{X}(\mathbb{R})$, and these sections induce the smooth remodeling process at $X$ (see Definition 19).

Roughly speaking, all the particles undergo a smooth remodeling ($\Omega_{X}(\mathbb{R})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$) and the variation at different points is also smooth ($\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$).

**Theorem 26.** *Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a smooth remodeling process if, and only if:*

*(i)* $\dim\left(A\Omega(\mathcal{C})^{T}_{\epsilon((t,X))}\right)$ *is constant with respect to* $(t,X)$*;*

*(ii)* $\dim\left(A\Omega_{X}(\mathbb{R})^{\sharp}_{(t,X)}\right) = 1$*, for all* $(t,X) \in \mathcal{C}$.

*Here, $A\Omega(\mathcal{C})^{T}_{\epsilon((t,X))}$ (respectively, $A\Omega_{X}(\mathbb{R})^{\sharp}_{t}$) is the fiber of $A\Omega(\mathcal{C})^{T}$ (respectively, $A\Omega_{X}(\mathbb{R})^{\sharp}$) at $\epsilon((t,X))$ (respectively, $t$).*

To prove this theorem we need an auxiliary lemma.

**Lemma 27.** *Let $M$ be a manifold and a path-connected subset $X$ of $M$. Consider a regular foliation $\mathcal{F}$ of $M$ such that*

*(i) $X$ is a union of leaves of $\mathcal{F}$.*
*(ii) $X$ is not a leaf of $\mathcal{F}$.*

*Then, there exists a strictly coarser (singular) foliation of $M$ satisfying (i) in Theorem 26.*

**Proof.** Assume that $X$ is not a leaf of $\mathcal{F}$. Let $\varphi = \left(y^1, \ldots, y^n\right)$ be a foliation in a neighborhood $U$ of $x \in M$,

$$U := \{-\epsilon < y^1 < \epsilon, \ldots, -\epsilon < y^n < \epsilon\}, \tag{48}$$

such that the $k$-dimensional disk $\{y^{k+1} = \cdots = y^n = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}(x)$ with $U$ which contains $x$, and each $k$-dimensional disk $\{y^{k+1} = c_{k+1}, \ldots, y^n = c_n\}$, where $c_{k+1}, \ldots, c_n$ are constants, coincides with the path-connected component of the intersection of some $\mathcal{F}(y)$ with $U$. We may shrink $\epsilon$ enough to get that $U \cap X$ is path-connected.

Let $y$ be a point in $U \cap X$ which is not contained in $\mathcal{F}(x)$ (i.e., $\mathcal{F}(y) \neq \mathcal{F}(x)$). Then, there exists a differentiable path $\alpha : I \to U \cap X$, with $I = [0, 1]$, such that

$$\alpha(0) = x, \quad \alpha(1) = y.$$

Then, we consider

$$\mathcal{C} := \{z \in M \ : \ \mathcal{F}(z) \cap \overline{\alpha}(0, 1) \neq \emptyset\}. \tag{49}$$

Consider the path $\overline{\alpha} : I \to \mathcal{U}, \mathcal{U} = \varphi^{-1}(U \cap X)$, given by

$$\overline{\alpha} = \varphi^{-1} \circ \alpha.$$

Then, by using the local expression (48),

$$\mathcal{C} \cap U : \{-\epsilon < y^1 < \epsilon, \ldots, -\epsilon < y^k < \epsilon, y^{k+1} = \alpha^{k+1}(t), \ldots, y^{k+1} = \alpha^n(t)\}_{t \in (0,1)}, \tag{50}$$

where $\alpha^i$ are the coordinates of $\alpha$ respect to $\varphi$. Using the rank theorem we may transform $\varphi$ to get that

$$\mathcal{C} \cap U : \{-\epsilon < y^1 < \epsilon, \ldots, -\epsilon < y^k < \epsilon, y^{k+1} = t, 0, \ldots, 0)\}_{t \in (0,1)}. \tag{51}$$

Consider the foliation $\mathcal{G}$ of $M$ such that:

•$\mathcal{G}(z) = \mathcal{F}(z)$ for each $z \notin \mathcal{C}$;
•$\mathcal{G}(z) = \mathcal{C}$ for each $z \in \mathcal{C}$.

Obviously, $\mathcal{G}$ is a strictly coarser division of $M$ and satisfies (i). Furthermore, it is an easy exercise to prove that $\mathcal{G}$ is a singular foliation (see Equation (51)). □

By separating within path-connected components we may prove the following result.

**Lemma 28.** *Let $M$ be a manifold and let $X$ be a subset of $M$. Consider a regular foliation $\mathcal{F}$ of $M$ such that:*

*(i) $X$ is a union of leaves of $\mathcal{F}$;*
*(ii) there is at least a path-connected component of $X$ which is not a leaf of $\mathcal{F}$.*

*Then, there exists a strictly coarser (singular) foliation of $M$ satisfying (i).*

Thus, roughly speaking, for each manifold $M$ and any subset $X$ which is not a submanifold of $M$, the maximal foliation satisfying (i) is necessarily singular.

**Proof of Proposition 26.** Note that condition (ii) is equivalent to the fact that all the $X$-material groupoids $\Omega_X(\mathbb{R})$ are transitive Lie subgroupoids of $\Phi(\mathcal{V})$. Thus, we only have to deal with condition *i*), i.e.,

$$\dim\left(A\Omega(\mathcal{C})^T_{\epsilon((t,X))}\right),$$

is constant with respect to $(t, X)$. Then, the material foliation $\overline{\mathcal{F}}$ is regular.

On the one hand, $\overline{\mathcal{F}}$ is a maximal foliation whose leaves are contained in the $\overline{\beta}$-fibers (see Corollary 2). Then, taking into account Lemma 28, the path-connected components of the $\overline{\beta}$-fibers of $\Omega(\mathcal{C})$ have to be leaves of the foliation. Finally, the local charts of the transitive Lie subgroupoids $\Omega\left(\overline{\mathcal{F}}(x)\right)$ (see the proof of Corollary 21) define a structure on $\Omega(\mathcal{C})$ of Lie subgroupoid of $\Phi(\mathcal{V})$. □

Hence, Proposition 26 provides us with a computational way of dealing with the smooth remodeling processes. In other words, by Equations (25) and (33), $\mathcal{C}$ represents a smooth remodeling if, and only if, the space of solutions of the equation,

$$\lambda \frac{\partial W}{\partial t} + \Theta^i \frac{\partial W}{\partial x^i} + \Theta^i_j \frac{\partial W}{\partial y^j_i} = 0, \tag{52}$$

where $\lambda$, $\Theta^i$, and $\Theta^i_j$ are functions depending on $t$ and $X$, has constant dimension and there exists a solution of

$$\lambda \frac{\partial W_X}{\partial t} + \Theta^i_j \frac{\partial W_X}{\partial y^j_i} = 0 \tag{53}$$

with $\lambda \neq 0$. Note that if Equation (53) is satisfied, the space of solutions of Equation (52) has, at least, dimension one.

## 6. Aging

**Definition 22.** Let $\mathcal{C}$ be a body-time manifold. A material particle $X \in \mathcal{B}$ undergoes *aging* when it is not undergoing a remodeling, i.e., not all the instants are connected by a material isomorphism. $\mathcal{C}$ is a *process of aging* if it is not a process of remodeling.

Clearly, if the material response is not preserved through time via material isomorphisms, the intrinsic constitutive properties are changing with time. Although it appears to be natural, there is not a proper definition of *smooth aging*. The presentation of this definition is another of the contributions of this paper (Definition 23).

**Proposition 29.** *Let $\mathcal{C}$ be a body-time manifold. A material particle $X \in \mathcal{B}$ undergoes aging if, and only if, the $X$-material groupoid $\Omega_X(\mathbb{R})$ is not transitive. Here $\mathcal{C}$ represents aging if, and only if, for some material point $X$, the $X$-material groupoid $\Omega_X(\mathbb{R})$ is not transitive.*

**Corollary 30.** *Let $\mathcal{C}$ be a body-time manifold with a uniform state. Here $\mathcal{C}$ represents aging if, and only if, the material groupoid $\Omega(\mathcal{C})$ is not transitive.*

We are ready to present a definition of *smooth aging*.

By analogy with smooth remodeling, to define *smooth aging* of the global body-time manifold as the smooth aging of all the material particles is not enough. We need also to impose smoothness on the variation along the material particles.

**Definition 23.** Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents *smooth aging* if $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$, and there is a particle $X$ such that the $X$-material groupoid $\Omega_X(\mathbb{R})$ is a not transitive Lie subgroupoid of $\Phi(\mathcal{V})$.

In other words, $\mathcal{C}$ represents smooth aging if the variation of the body is "*smooth*" ($\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$) and it is not a smooth remodeling.

Observe that, taking into account Proposition 15, if $\Omega(\mathcal{C})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$, then for all particles $X$, the $X$-material groupoid $\Omega_X(\mathbb{R})$ is a Lie subgroupoid of $\Phi(\mathcal{V})$. Thus, the only imposition is given by the lack of transitivity of some $X$-material groupoids.

Consider $\Omega(\mathcal{C})$, the material groupoid associated with the body-time manifold $\mathcal{C}$. Then, we have available the material distribution $A\Omega(\mathcal{C})$, the body-material distribution $A\Omega(\mathcal{C})^\sharp$, and the uniform-material distribution $A\Omega(\mathcal{C})^B$, and their associated foliations, the material foliation $\overline{\mathcal{F}}$, body-material foliation $\mathcal{F}$, and uniform-material foliation $\mathcal{G}$, respectively.

**Proposition 31.** *Let $\mathcal{C}$ be a body-time manifold. Here $\mathcal{C}$ represents a smooth aging process if, and only if:*

*(i)* $\dim\left(A\Omega\left(\mathcal{C}\right)^{T}_{\epsilon((t,X))}\right)$ *is constant with respect to* $(t,X)$*;*
*(ii)* *For some* $X$*,* $\dim\left(A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}\right)=0$*, for some* $t$*.*

*Here,* $A\Omega\left(\mathcal{C}\right)^{T}_{\epsilon((t,X))}$ *(respectively,* $A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}$*) is the fiber of* $A\Omega\left(\mathcal{C}\right)^{T}$ *(respectively,* $A\Omega_X\left(\mathbb{R}\right)^{\sharp}$*) at* $\epsilon\left((t,X)\right)$ *(respectively,* $t$*).*

**Proof.** Note that if $\Omega\left(\mathcal{C}\right)$ is a Lie subgroupoid of $\Phi\left(\mathcal{V}\right)$, then, by Proposition 15, all the $X$-material groupoids are Lie subgroupoids of $\Phi\left(\mathcal{V}\right)$. Then, $\dim\left(A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}\right)$ does not depend on time at any instant $t$. Therefore, $\Omega_X\left(\mathbb{R}\right)$ is not transitive if, and only if, $\dim\left(A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}\right)=0$, for all $t$. Then, the proof is analogous to Theorem 26. □

In this way, again, we present a result characterizing smooth aging of the material evolution which gives a computational way of testing it.

**Definition 24.** Let $\mathcal{C}$ be a body-time manifold. The body $\mathcal{B}$ is said to undergo a *uniform aging* if for each $t\in\mathbb{R}$ all the points $(t,X)\in\mathcal{C}$ are isomorphic and it is not undergoing a uniform remodeling.

Intuitively, in a process of uniform aging the material properties change equally at all points.

**Proposition 32.** *Let* $\mathcal{C}$ *be a body-time manifold. The body* $\mathcal{B}$ *undergoes uniform aging if, and only if, for all* $t$*, the* $t$*-material groupoid* $\Omega_t\left(\mathcal{B}\right)$ *is transitive and the material groupoid* $\Omega\left(\mathcal{C}\right)$ *is not transitive.*

Thus, a process of aging is uniform if all the states of the body are uniform but the intrinsic properties of the body vary along time.

**Proposition 33.** *Let* $\mathcal{C}$ *be a body-time manifold. Then* $\mathcal{C}$ *presents a smooth uniform aging process if, and only if:*

*(i)* $\dim\left(A\Omega\left(\mathcal{C}\right)^{T}_{\epsilon((t,X))}\right)$ *is constant with respect to* $(t,X)$*;*
*(ii)* *for some* $X$ *and* $t$*,* $\dim\left(A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}\right)=0$*;*
*(iii)* *for all* $t$ *and some* $X$*,* $\dim\left(A\Omega_t\left(\mathcal{C}\right)^{\sharp}_{X}\right)=3$*.*

*Note that, by Proposition 15,* $\dim\left(A\Omega_X\left(\mathbb{R}\right)^{\sharp}_{t}\right)$ *and* $\dim\left(A\Omega_t\left(\mathcal{C}\right)^{\sharp}_{X}\right)$ *are constant on* $t$ *and* $X$*, respectively.*

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## ORCID iDs

Víctor Manuel https://orcid.org/0000-0003-3135-3303
Marcelo Epstein https://orcid.org/0000-0003-1216-0039

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